Border basis: a useful tool for constructions in Algebra

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(from joints paper with H. Lombardi and, Brachat, Mourrain)

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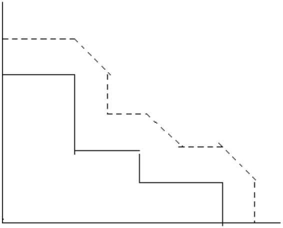
. The $h_\alpha^{\mathbf{z}}(\underline{\mathbf{x}})$ will be called, **the border relations** of $\mathcal{A}$ w.r.t. B.



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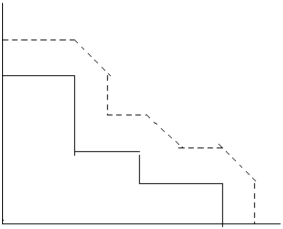

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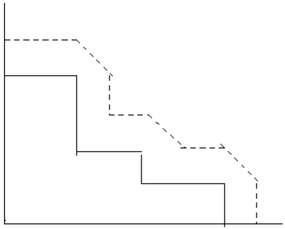

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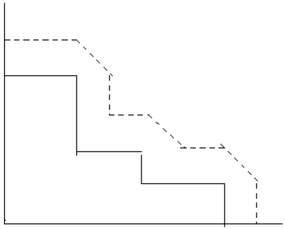
- The tables of multiplication $M^{\mathbf{z}}_{x_i} : \langle B \rangle \to \langle B \rangle$ are constructed using $M^{\mathbf{z}}_{x_i}(\underline{\mathbf{x}}^{\beta}) = N^{\mathbf{z}}(x_i \underline{\mathbf{x}}^{\beta})$ for $\beta \in B$.



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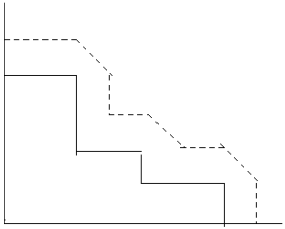

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- Notice that the coefficients of the matrix of $M^{\mathbf{z}}_{x_i}$ in the basis B are linear in the coefficients $\mathbf{z}$'s.

Border equations

- Conversely, if we are interested in characterizing the coefficients $\mathbf{z} := (z_{\alpha,\beta})_{\alpha\in\partial B,\beta\in B}$ such that the polynomials $(h_{\alpha}^{\mathbf{z}}(\underline{\mathbf{x}}))_{\alpha\in B}$ are the border relations of some free A-algebra $\mathcal{A}^{\mathbf{z}} = A[x_1,\dots,x_n]/I$ with basis B.

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Theorem

Let B be a set of μ monomials connected to 1. The polynomials $h^{\mathbf{z}}_{\alpha}(\underline{\mathbf{x}})$ are the border relations of some free quotient algebra $\mathcal{A}^{\mathbf{z}}$ of $A[x_1,\ldots,x_n]$ of basis B iff

$$M^{\mathbf{z}}_{x_i}\circ M^{\mathbf{z}}_{x_j} - M^{\mathbf{z}}_{x_j}\circ M^{\mathbf{z}}_{x_i} = 0 \quad \text{for} \quad 1\leqslant i<j\leqslant n. \tag{1}$$

$$\mathcal{H}_B := \{\mathbf{z} = (z_{\alpha,\beta}) \in \mathbb{K}^{\partial B\times B};\, M^{\mathbf{z}}_{x_i}\circ M^{\mathbf{z}}_{x_j} - M^{\mathbf{z}}_{x_j}\circ M^{\mathbf{z}}_{x_i} = 0 \ {}_{1\leqslant i<j\leqslant n}\}$$

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"isolated, embedded points, points going to infinite"

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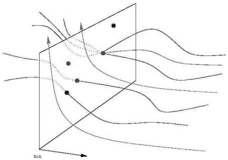

Flatness means the monomial basis B is still a basis of $\mathcal{A}$ as $\mathbb{K}[[\varepsilon]]$ module (assumed $\mathcal{A}$ is finite $\mathbb{K}[[\varepsilon]]$– module)

Flatness criterion

- More generally, let $(A, \mathfrak{m}, \mathbb{K})$ be a henselian ring. Start with a deformed situation $\mathbf{f} \in A[\mathbf{x}]^s$, $\mathbf{f} = \mathbf{f}^0 + \varepsilon \mathbf{f}^1 + \cdots$; $\varepsilon \in \mathfrak{m}$, denote by $\mathrm{I} = (\mathbf{f})\mathrm{A}[\mathbf{x}]$, $\mathrm{I}^0 = (\mathbf{f}^0)\mathbb{K}[\mathbf{x}]$ and $\mathcal{A} := A[\mathbf{x}]/\mathrm{I}$ and the residual (initial) situation $\mathcal{A}^0 = \mathbb{K}[\mathbf{x}]/\mathrm{I}^0$.

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QUESTION:

Conditions for $\mathcal{A} = A[\mathbf{x}]/\mathrm{I}$ (resp. $\mathcal{A}_a = S^{-1}(A[\mathbf{x}]/\mathrm{I})$ to be a flat (hence free) A module? What can we say of a border basis of $\mathcal{A}$ (or $\mathcal{A}_a$), assuming one knows a border basis mod. $\mathfrak{m}$?

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• We proof a *Bezout local theorem* for A a henselian equicharact. ring : which is elementary (and constructive for DVR.)

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Using the henselianity, **we lift them to get border relations in $\mathcal{A}$**

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WE GET FLATNESS, iff the lifted border relations :

- i) verify the equations of commutativity, in order to be border basis of $A[\mathbf{x}]/(h_{\alpha\beta})$, and
- ii) generate the ideal of the beginning: $\mathrm{I}\ \mathrm{S}^{-1}\mathrm{A}[\mathbf{x}] = \mathcal{H}$

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- Plücker coordinates of Δ as an element of $\mathbb{P}(\wedge^{\mu} S_d^*)$ are given by:

$$\Delta_{\beta_1,\ldots,\beta_\mu} = \begin{vmatrix} \delta_1(\mathbf{x}^{\beta_1}) & \cdots & \delta_1(\mathbf{x}^{\beta_\mu}) \\ \vdots & & \vdots \\ \delta_\mu(\mathbf{x}^{\beta_1}) & \cdots & \delta_\mu(\mathbf{x}^{\beta_\mu}) \end{vmatrix}$$

for $\beta_i \in \mathbb{N}^{n+1}$, $|\beta_i| = d$ and $\beta_1 < \cdots < \beta_\mu$.

The $\mathrm{Hilb}^{\mu}(\mathbb{P}^n)$ inside the $\mathbf{Gr}^{\mu}_{S_d^*}(X)$

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- In A. -Brachat- Mourrain (2008), we find an inmersion of $\mathrm{Hilb}^{\mu}(\mathbb{P}^n)$, inside the $\mathbf{Gr}^{\mu}_{S_d^*}(X)$ with global equations of deree two.

In the following we show how to get $\mathrm{Hilb}^{\mu}(\mathbb{P}^n)$ inside a product of Grasmanians with equations of degree two.

Global equations for $\mathrm{Hilb}^{\mu}(\mathbb{P}^{n})$

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$B = (b_1, \ldots, b_\mu)$ be a family of homogeneous polynomials of degree d, then,
$\Delta_B\, a - \sum_{i=1}^{\mu} \Delta_{B^{[b_i|a]}}\, b_i = 0 \text{ in } \Delta, \text{for } a \in S_d^A$
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where $B^{[b_i|a]} = (b_1, \ldots, b_{i-1}, a, b_{i+1}, \ldots, b_\mu)$. Let it be

$$M := \begin{bmatrix} \delta_1(a) & \delta_1(b_1) & \cdots & \delta_1(b_\mu) \\ \vdots & & & \vdots \\ \delta_\mu(a) & \delta_\mu(b_1) & \cdots & \delta_\mu(b_\mu) \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

and develop its determinant along the last row of M. We get the last equality of rows in the identity below. The others come from developping a deteminant with a repetead row.

$$M \begin{bmatrix} \Delta_B \\ \Delta_{B^{[b_1|a]}} \\ \vdots \\ \Delta_{B^{[b_\mu|a]}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \det(M) \end{bmatrix}.$$

We conclude that $\Delta_B\, a - \sum_{i=1}^{\mu} \Delta_{B^{[b_i|a]}}\, b_i = 0$ in Δ since all δ_j vanishes at this element, and they are a basis of Δ^*.

Conti.

- **Theorem:** Let $d \geq \mu$ be an integer. $\mathbf{Hilb}^{\mu}_{\mathbb{P}^n}(X)$ is the projection on $\mathbf{Gr}^{\mu}_{S^*_d}(X)$ of the variety of $\mathbf{Gr}^{\mu}_{S^*_d}(X) \times \mathbf{Gr}^{\mu}_{S^*_{d+1}}(X)$ defined by the equations

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for all families B (resp. B') of μ (resp. $\mu - 1$) monomials of degree d (resp. $d+1$), all monomial $a \in S^A_d$ and for every k (where $B', x_k a$ is the family $(b'_1, \ldots, b'_{\mu-1}, x_k a)$.

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THANK YOU FOR YOUR ATTENTION!