Budan Tables and Virtual Roots of Real Polynomials

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Fête de Henri LOMBARDI
Besançon, Octobre, 2011.

• After the works of Gonzales-Vega, Lombardi, Mahé, [GLM:1998], and Coste, Lajous, Lombardi, Roy, [CLLR:2005], we revisit the concept of virtual roots of a univariate polynomial f with real coefficients.

• We show, in the generic case, how to locate the virtual roots of f on the Budan table.

• We consider a property $(\mathcal{P})$ of a polynomial f, which is generically satisfied, it eases the topological-combinatorial description and study of the Budan tables.

- We study the topology of the positive (resp. negative) blocks components of a Budan table, and characterize the virtuals roots using connected blocks components.

- A natural extension of the information collected by the virtual roots provides alternative representations of $(\mathcal{P})$-polynomials

- An attached tree structure allows a finite stratification of the space of $(\mathcal{P})$-polynomials.

DEFINITION[Budan table]:

Let f be a monic univariate polynomial of degree n.
The Budan table of f is the union of $n+1$ infinite rectangles of height one $L_i := \mathbb{R} \times [i - 1/2, i + 1/2[$ for i from 0 to n, called rows.
For i from 0 to n, each row L_i is the union of a set of open rectangles (possibly infinite), separated by vertical segments.
We color in black the rectangles corresponding to negative values of the $(n-i)$-th derivative $f^{(n-i)}$ of f, and we color in gray the rectangles corresponding to positive values of $f^{(n-i)}$.

Figure: A Budan table of degree 6

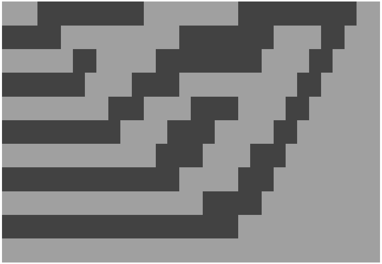


Figure: A Budan table of of degree 10

If b is a root of multiplicity k of f with $k \leq n$ then for sufficiently small positive h, denoting by s the sign of $f^{(k)}(b)$, the columns of the Budan table of f near b are shown.
Similarly if c is a root of multiplicity k of $f^{(m)}$.

	b-h	b	b+h			c-h	c	c+h
$\text{sgn}(f)$	$(-1)^k s$	0	s		$\text{sgn}(f^{(m-1)})$	s_1	s_1	s_1
$\text{sgn}(f')$	$(-1)^{k+1} s$	0	s		$\text{sgn}(f^{(m)})$	$(-1)^k s_2$	0	s_2
...	...	0	s		...	...	0	s_2
$\text{sgn}(f^{(k-1)})$	$-s$	0	s		$\text{sgn}(f^{(m+k-1)})$	$-s_2$	0	s_2
$\text{sgn}(f^{(k)})$	s	s	s		$\text{sgn}(f^{(m+k)})$	s_2	s_2	s_2

• The figure shows that when x moves in $\mathbf{R}$, the signs in the columns of a Budan table are continuous on the right.

• A classical descriptor attached to a Budan table is the function $V_f(x)$ of the real indeterminate x with values in the set of integers $\mathbf{N}$, it counts the number of sign changes in the sequence formed by f and its derivatives evaluated at x.

• The Budan table is a finer invariant than V_f attached to the polynomial f.

DEFINITION[($\mathcal{P}$)-polynomials]:

A polynomial in $\mathbf{R}[x]$ satisfies condition $(\mathcal{P})$ if and only if:
each derivative of g has simple roots, and all these roots are two by two distinct.
A monic polynomial satisfying this condition will be called a $(\mathcal{P})$-polynomial.
The property $(\mathcal{P})$ is generically satisfied.
Now on, if not specified we will assume that f satisfies condition $(\mathcal{P})$.

THEOREM

Let f be a $(\mathcal{P})$-polynomial of degree n, and let $m \leq n$ be the number or real (simple) roots of f. Then m and n have the same parity, $n = m + 2p$ and the Budan table of f is a $\mathcal{GB}$ table of degree n.

DEFINITION[Generic Budan table]:

A table B with $(n+1)$ rows L_i formed by rectangles of alternating colors, is a $\mathcal{GB}$ table of degree n iff:

- The row L_0 is a gray, the infinite rightest rectangle of each row is gray, The first (infinite) rectangle of each row L_i is alternatively gray or black.
- If i is even (resp. odd) the number of rectangles on the row L_i is even (resp. odd).

- Let $(l+1)$ be the number of rectangles of the top row L_n, then $l \leq n$ and $n-l$ is an even number $2p$.
There are $l+p+1$ same-color-connected components of B.
Each non first rectangle of L_i, $i > 0$ is connected on the left to a rectangle of the same color of the row L_{i-1}.
The l first rectangles of L_n are in separated same-color-connected components.
The p other connected components, bounded on the right, are surrounded by connected components of the opposite color.
- The previous item is true, replacing n by any m, $0 < m < n$, and B by the table formed by the lower $m+1$ rows.

DEFINITION[Virtual roots]:

Let f be a $(\mathcal{P})$-polynomial of degree n. The x value of the rightest upper segment of a connected component (either gray or black) of the Budan table of f is called a virtual root of f.

Any real root (in the usual sense) of f is a virtual root. Let $m \leq n$ be the number or real (simple) roots of f, and let $n - m = 2p$. There are p virtual non real roots of f, we say that there are of multiplicity two; each of them is a root of some derivative of f of positive order.

Figure: A Budan table of degree 6

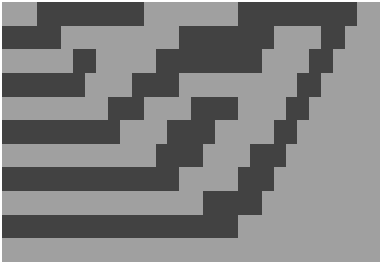


Figure: A Budan table of of degree 10

DEFINITION[Augmented Virtual roots]:

We call augmented virtual root of f the pair (y, k) formed by a virtual root of f and the order of the derivative of f which vanishes at y, i.e. $f^{(k)}(y) = 0$.

The augmented virtual roots of f only depend on the Budan table BT of f.

PROPOSITION

Let f be a $(\mathcal{P})$-polynomial of degree n.
By Rolle's theorem between two successive roots $a < b$ of some derivative $f^{(m)}$ with $0 \leq m \leq n-2$, (or in $\mathbf{R}$ if $f^{(m)}$ has no root), there is an odd number $2r+1$ of roots $(X_1 < ... < X_{2r+1})$ of the next derivative $f^{(m+1)}$.

Then the r roots with an even index $(X_2, ...X_{2r})$ are virtual non real roots of f. (Respectively in the infinite interval.)

For each augmented virtual non real root (y, k) of f, we have

$$f^{(k-1)}(y)f^{(k+1)}(y) > 0.$$

DEFINITION[$\mathcal{S}(f)$]:

We denote by $\mathcal{S}(f)$ the system of $n = m + 2p$ data formed by:
the m real roots $(x_i), 1 \leq i \leq m$, of f;
the p augmented virtual root $(y_j, k_j),\ 1 \leq j \leq p$ with $k > 0$
and the p corresponding values $w_j := f^{(k-1)}(y_j), 1 \leq j \leq p$.

PROPOSITION:

Two different $(\mathcal{P})$-polynomials f and g define different systems $\mathcal{S}(f) \neq \mathcal{S}(g)$

It corresponds to a so-called homogeneous Hermite-Birkhoff interpolation problem.

QUESTIONS:

Q1 Given a system $\mathcal{S}$ as above, does there exist a $(\mathcal{P})$-polynomial f which satisfies these data ?
Q2 Let B be a $\mathcal{GB}$ table of degree n, does there exist a $(\mathcal{P})$-polynomial f of degree n, which admits B as its Budan table?
The general answers is NO !

PROPOSITION:

The transposed incidence matrix E of a $(\mathcal{P})$-polynomial f only depends on the augmented virtual roots (y_i, k_i) of f.
It is the $(n, m+p)-$matrix $E = (e_{j,i})$ such that n of its entries are 1, and the others are 0 according to the following rule:

1. If $k_i{=}0$ then $e_{0,i} = 1$.
2. If $k_i > 0$ then $e_{k_i,i} = 1$ and $e_{k_i-1,i} = 1$.
3. Otherwise $e_{j,i} = 0$.

HINT: It is proved that a homogeneous HB problem whose incidence matrix is both conservative and satisfies Polya condition, has only the trivial solution.

The previously defined matrix E (attached to $m+p$ augmented virtual roots) is conservative and satisfies Polya condition.

There exist a unique monic polynomial F satisfying the vanishing conditions :
$F(x_i) = 0, 1 \leq j \leq m+$, $F^{(k_i)}(y_i) = 0, 1 \leq j \leq p$,
$w_l := F^{(k_i - 1)}(y_i), 1 \leq i \leq p$.

DEFINITION[Budan tree]:

In the Budan table of $(\mathcal{P})$-polynomial f, we replace the list of the augmented virtual roots (y_i, k_i), by the corresponding list of pairs of integers (i, k_i). with $1 \leq i \leq m + p$ to form the nodes of a bi color tree with $n + 1$ leaves.
The edges are obtained by contracting the same-color-connected components.

We say that the Budan tree is decorated if the coordinates (i, k_i), $1 \leq i \leq m + p$ of its nodes are given.

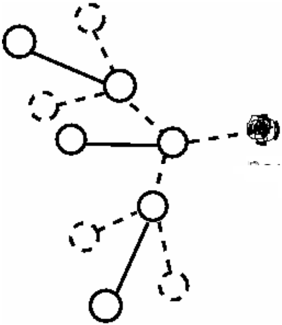


Figure: A Budan tree of degree 6

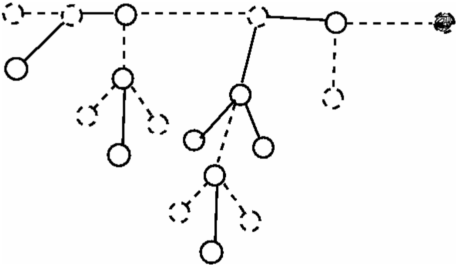


Figure: A Budan tree of degree 10

QUESTION:

Q3 Let B be a $\mathcal{GB}$ table of degree n, does there exist a $(\mathcal{P})$-polynomial f of degree n, which admits the same Budan tree than B ?

DEFINITION[Stratifications]:

Let T denote a decorated Budan tree, denote by $\mathcal{T}$ the non decorated tree.

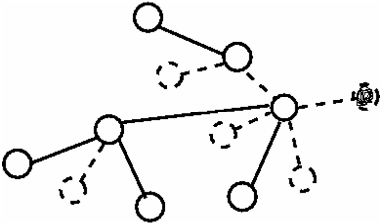
We consider the stratum Σ_T (resp. $\Theta_{\mathcal{T}}$) formed by all the $(\mathcal{P})$-polynomials f such that $T(f) = T$ (resp. $\Theta_{\mathcal{T}}(f) = \mathcal{T}$).

PROPOSITION:

The stratum Σ_T and $\Theta_{\mathcal{T}}$ are semi-algebraic sets.
The virtual roots of a the polynomials f in a stratum Σ_T depend analytically on the coefficients of f.

QUESTION:

Q4 Are the strata Σ_T connected in the set of $(\mathcal{P})$-polynomials ?



The previous constructions can be extended to non-generic polynomials.

Let us now quote a continuity result of
Gonzales-Vega, Lombardi, Mahé, [GLM:1998], and
Coste, Lajous, Lombardi, Roy, [CLLR:2005].

THEOREM:
The virtual roots of a monic polynomial f depend continuously on the coefficients of f.

There are different possible extensions of our work:

Truncated Budan tables.

Fewnomials.

"Circular" differentiation.
Replace the input polynomial $f(x)$ of degree n by its homogenization $F(X, Y)$ in degree n,
and then set $X = cos(t)$, $Y = sin(t)$ to get a trigonometric polynomial $G(t)$ depending on n coefficients.

BONNE

FETE

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