Generalized geometric theories and set-generated classes

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The constructive set theory, CZF

The constructive Zermelo-Fraenkel set theory, CZF (Aczel, 1978)

- has a quite natural interpretation in the Martin-Löf type theory
- is a predicative theory
  - without power set axiom
  - without full separation axiom
  - with restricted separation axiom

CZF

The axioms and rules of **CZF** are the axioms and rules of intuitionistic predicate logic with equality, and the following set theoretic axioms:

- Extensionality: $\forall a \forall b(\forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b)$.
- Pairing: $\forall a \forall b \exists c \forall x(x \in c \leftrightarrow x = a \vee x = b)$.
- Union: $\forall a \exists b \forall x(x \in b \leftrightarrow \exists y \in a(x \in y))$.
- Restricted Separation:

$$\forall a \exists b \forall x(x \in b \leftrightarrow x \in a \wedge \varphi(x))$$

  for every *restricted* formula $\varphi(x)$, where a formula $\varphi(x)$ is restricted, or Δ_0, if all the quantifiers occurring in it are bounded, i.e. of the form $\forall x \in c$ or $\exists x \in c$.

CZF

- Strong Collection:

$$\forall a(\forall x \in a \exists y \varphi(x,y) \rightarrow \exists b(\forall x \in a \exists y \in b \varphi(x,y) \wedge \forall y \in b \exists x \in a \varphi(x,y)))$$

  for every formula $\varphi(x,y)$.

- Subset Collection:

$$\forall a \forall b \exists c \forall u(\forall x \in a \exists y \in b \varphi(x,y,u) \rightarrow \exists d \in c(\forall x \in a \exists y \in d \varphi(x,y,u) \wedge \forall y \in d \exists x \in a \varphi(x,y,u)))$$

  for every formula $\varphi(x,y,u)$.

CZF

- Infinity:

$$
\begin{array}{ll}
(\mathbf{N}1) & 0 \in \mathbf{N} \wedge \forall x(x \in \mathbf{N} \rightarrow x+1 \in \mathbf{N}), \\
(\mathbf{N}2) & \forall y(0 \in y \wedge \forall x(x \in y \rightarrow x+1 \in y) \rightarrow \mathbf{N} \subseteq y),
\end{array}
$$

  where $x+1$ is $x \cup \{x\}$, and 0 is the empty set $\emptyset$.
- $\in$-Induction:

$$
(\mathrm{IND}_{\in}) \qquad \forall a(\forall x \in a \varphi(x) \rightarrow \varphi(a)) \rightarrow \forall a \varphi(a)
$$

  for every formula $\varphi(a)$.

CZF

- For each formula φ, the collection $\{x \mid \varphi(x)\}$ is a *class*.
  - $\{x \mid x = x\}$ is a class;
  - $\{x \mid x \subseteq y\}$ is a class.

- A class C is a set if $\exists x \forall y (y \in C \leftrightarrow y \in x)$.

CZF

- The class of total relations between a and b is denoted by $\mathrm{mv}(a,b)$:

$$r \in \mathrm{mv}(a,b) \Leftrightarrow r \subseteq a \times b \wedge \forall x \in a \exists y \in b((x,y) \in r).$$

- The class of functions from a to b is denoted by b^a:

$$\begin{aligned} f \in b^a \Leftrightarrow\ & f \in \mathrm{mv}(a,b) \\ & \wedge \forall x \in a \forall y, z \in b((x,y) \in f \wedge (x,z) \in f \rightarrow y = z). \end{aligned}$$

CZF

In **CZF**, we can prove

- Fullness:
  $\forall a \forall b \exists c (c \subseteq \mathrm{mv}(a,b) \wedge \forall r \in \mathrm{mv}(a,b) \exists s \in c (s \subseteq r))$,

and, as a corollary, we see that b^a is a set, that is

- Exponentiation: $\forall a \forall b \exists c \forall f (f \in c \leftrightarrow f \in b^a)$.

Set-generated classes

A class X of subsets of a set S is *set-generated* if there exists a subset G of X such that

$$\alpha = \bigcup\{\beta \in G \mid \beta \subseteq \alpha\}$$

for each $\alpha \in X$. We call the set G a *generating subset* of the class X.

- The class $\mathrm{Pow}(S)$ of opens of the discrete topology on a set S is not a set in **CZF**.
- A base $\{\{s\} \mid s \in S\}$ of the opens $\mathrm{Pow}(S)$ is a set in **CZF**.
- Note that $\mathrm{Pow}(S)$ is a set-generated class with a generating subset $\{\{s\} \mid s \in S\}$.

Set-generated classes

Proposition

Let X be a class of inhabited subsets of a set S, and let $\mathrm{Min}(X)$ be a class of minimal elements of X, that is,

$$\mathrm{Min}(X) = \{x \in X \mid \forall y \in X (y \subseteq x \to y = x)\}.$$

If X is set-generated, then $\mathrm{Min}(X)$ is a set.

Implications and theories

Definition

The *generalized geometric implications* (simply, implications) and *generalized geometric theories* (simply, theories) over a set S, and their rank, are defined simultaneously by

1. s is a implication of rank 0 for each $s \in S$;
2. if σ is a finite subset of S and Γ is a set of theories of rank n, then $\bigwedge \sigma \to \bigvee_{U \in \Gamma} \bigwedge U$ is a implication of rank $n + 1$;
3. a set T of implications of rank $\leq n$ is a theory of rank n.

Implications and theories

- $\bigvee_{U\in\Gamma}\bigwedge U \equiv \bigwedge\emptyset \to \bigvee_{U\in\Gamma}\bigwedge U$,
- $s \to \bigvee_{U\in\Gamma}\bigwedge U \equiv \bigwedge\{s\} \to \bigvee_{U\in\Gamma}\bigwedge U$,
- $\bigwedge\sigma \to \bigvee_{\varphi\in U}\varphi \equiv \bigwedge\sigma \to \bigvee_{\varphi\in U}\bigwedge\{\varphi\}$,
- $\bigwedge\sigma \to \bigwedge U \equiv \bigwedge\sigma \to \bigvee_{U\in\{U\}}\bigwedge U$.

Models of theories

For an implication $\varphi \equiv \bigwedge \sigma \to \bigvee_{U \in \Gamma} \bigwedge U$ of positive rank, we denote the sets σ and Γ by σ_φ and Γ_φ, respectively.

Definition

The relation $\models$ between a subset α of S, and implications s (of rank 0), φ (of positive rank) and a theory T over S is defined by

1. $\alpha \models s$ if $s \in \alpha$;
2. $\alpha \models \varphi$ if $\sigma_\varphi \subseteq \alpha$ implies $\alpha \models U$ for some $U \in \Gamma_\varphi$;
3. $\alpha \models T$ if $\alpha \models \theta$ for all $\theta \in T$.

We say that α is a *model* of a theory T if $\alpha \models T$. The class of models of T is denoted by $\mathfrak{M}(T)$.

Extensions

An *extension* S' of a set S is a set with an inclusion (i.e., an injection) $\iota : S \to S'$.

We can naturally extend the inclusion ι to an inclusion $\hat{\iota}$ from the implications and the theories over S into the implications and the theories over S' of same rank by

$$\begin{array}{rcl}\hat{\iota}(s) & = & \iota(s), \\ \hat{\iota}(\varphi) & = & \bigwedge \iota(\sigma_\varphi) \to \bigvee_{U \in \Gamma_\varphi} \bigwedge \hat{\iota}(U), \\ \hat{\iota}(T) & = & \{\hat{\iota}(\theta) \mid \theta \in T\},\end{array}$$

where s and φ are implications of rank 0 and of positive rank, respectively, and T is a theory.

Extensions

Lemma

Let T be a theory over S, and let S' be an extension of S with an inclusion ι. Then $\iota^{-1}(\alpha') \in \mathfrak{M}(T)$ if and only if $\alpha' \in \mathfrak{M}(\hat{\iota}(T))$ for each $\alpha' \in \mathrm{Pow}(S')$.

Extensions

Let S' be an extension of a set S with an inclusion ι.

- A theory T' over S' is an *extension* of a theory T over S if $\iota^{-1}(\alpha') \in \mathfrak{M}(T)$ for each $\alpha' \in \mathfrak{M}(T')$.
- An extension is *conservative* if for each $\alpha \in \mathfrak{M}(T)$ there exists $\alpha' \in \mathfrak{M}(T')$ such that $\alpha = \iota^{-1}(\alpha')$.

Note that the theory $\hat{\iota}(T)$ is a conservative extension of a theory T.

Rank reduction

Proposition

Each theory of rank $n+1$ ($n \geq 1$) has a conservative extension of rank n.

Proposition

Let T' be a conservative extension of a theory T. If the class $\mathfrak{M}(T')$ of models of the theory T' is set-generated, then the class $\mathfrak{M}(T)$ of models of the theory T is set-generated.

Regular extension axiom

- A set A is *regular* if it is *transitive*, i.e. $a \subseteq A$ for each $a \in A$, and for each $a \in A$ and $R \in \mathrm{mv}(a, A)$ there exists $b \in A$ such that

$$\forall x \in a \exists y \in b((x, y) \in R) \wedge \forall y \in b \exists x \in a((x, y) \in R).$$

- A set A is *union-closed* if $\bigcup a \in A$ for each $a \in A$.

uREA: Every set is a subset of a union-closed regular set.

Regular extension axiom

- A regular set A is RRS_2-*regular* if for each $A' \subseteq A$, $R \in \mathrm{mv}(A' \times A', A')$ and $a_0 \in A'$, there exists $A_0 \in A$ such that $a_0 \in A_0 \subseteq A'$ and $\forall x, y \in A_0 \exists z \in A_0(((x, y), z) \in R)$.

$\mathrm{RRS}_2\text{-uREA}$: Every set is a subset of a union-closed RRS_2-regular set.

Regular extension axiom

DC: If $\forall x \in a \exists y \in a \psi(x, y)$ and $b_0 \in a$, then there exists a function $f : \mathbf{N} \to a$ such that $f(0) = b_0$ and

$$\forall n \in \mathbf{N} \psi(f(n), f(n+1)).$$

Proposition

$\mathrm{uREA} + \mathrm{DC} \Rightarrow \mathrm{RRS}_2\text{-uREA}$.

Theorem

Assume $\mathrm{RRS}_2\text{-uREA}$. *Then the class* $\mathfrak{M}(T)$ *of models of a theory* T *of rank* 1 *is set-generated.*

Relativized dependent choice

Let ϕ and ψ be arbitrary formulas.

RDC: If $\forall x[\phi(x) \to \exists y(\phi(y) \land \psi(x, y))]$ and $\phi(b_0)$, then there exists a function f with domain $\mathbf{N}$ such that $f(0) = b_0$ and

$$\forall n \in \mathbf{N}[\phi(f(n)) \land \psi(f(n), f(n+1))].$$

Note that RDC implies DC.

Theorem

Assume RDC. *Then the class* $\mathfrak{M}(T)$ *of models of a theory* T *of rank* 1 *is set-generated.*

Main result

Theorem

Assume RRS_2-uREA *or* RDC. *Then the class* $\mathfrak{M}(T)$ *of models of a theory* T *of rank* n *is set-generated.*

Algebra

Let $(R, +, \cdot, -, 0, 1)$ be a commutative ring.

- A subset I of R is an *ideal* I if
  1. $0 \in I$,
  2. $x, y \in I \Rightarrow x - y \in I$,
  3. $x \in R, y \in I \Rightarrow x \cdot y \in I$.

Proposition

Assume RRS_2-uREA *or* RDC. *Then the class of ideals is set-generated.*

Proof.

Note that the class of ideals is the class of models of the theory:

$$\begin{aligned} & \{0\} \cup \{\textstyle\bigwedge\{x, y\} \to x - y \mid x, y \in R\} \\ \cup \quad & \{y \to x \cdot y \mid x, y \in R\}. \end{aligned}$$

□

Algebra

- An ideal I is *nontrivial* if there is $x \in I$ with $\neg(x = 0)$.

Proposition

Assume RRS_2-uREA *or* RDC. *Then the class of minimal nontrivial ideals is a set.*

Proof.

Note that the class of nontrivial ideals is the class of models of the theory:

$$
\begin{aligned}
& \{0\} \cup \{\textstyle\bigvee_{x \in \{x \in R \mid \neg(x=0)\}} x\} \\
\cup \quad & \{\textstyle\bigwedge\{x, y\} \to x - y \mid x, y \in R\} \\
\cup \quad & \{y \to x \cdot y \mid x, y \in R\}.
\end{aligned}
$$

□

Neighbourhood space

- A *neighbourhood space* is a pair (X, τ) consisting of a set X and a subset τ of $\mathrm{Pow}(X)$ such that
  1. $\forall x \in X \exists U \in \tau (x \in U)$,
  2. $\forall x \in X \forall U, V \in \tau [x \in U \cap V \to \exists W \in \tau (x \in W \subseteq U \cap V)]$.

  We say that τ is an *open base* on X.
- A subset A of X is *open* if for each $x \in A$ there exists $U \in \tau$ such that $x \in U \subseteq A$.
- A function f between neighbourhoos spaces (X, τ) and (Y, σ) is *continuous* if $f^{-1}(V)$ is open for each $V \in \sigma$.

Neighbourhood space

Let X be a set.
Let $\{(X_i, \tau_i) \mid i \in I\}$ be a family of neighbourhood spaces, and let $\{f_i : X_i \to X \mid i \in I\}$ be a family of functions.

- An open base τ on X is *final* for the family $\{f_i \mid i \in I\}$ if for any neighbourhood space (Y, σ) and any function $g : X \to Y$,

  g is continuous $\Leftrightarrow g \circ f_i : X_i \to Y$ is continuous for each $i \in I$.

Neighbourhood space

Proposition

Assume RRS_2-uREA *or* RDC. *Then the class*

$$C = \{U \in \mathrm{Pow}(X) \mid f_i^{-1}(U) \text{ is open for each } i \in I\}$$

is set-generated, and the generating set is a final open base on X.

Proof.

Note that C is the class of models of the theory:

$$\{f_i(x) \to \bigvee_{x \in V \in \mathcal{T}_i} \bigwedge_{y \in V} f_i(y) \mid x \in X_i, i \in I\}.$$

□

Formal topology

- A *formal topology* $(S, \leq, \lhd)$ is a preordered set $(S, \leq)$ equipped with a subclass $\lhd \subseteq S \times \mathrm{Pow}(S)$ such that
  1. $a \in U \Rightarrow a \lhd U$,
  2. $a \lhd U$ and $\forall c \in U(c \lhd V) \Rightarrow a \lhd V$,
  3. $a \lhd U$ and $a \lhd V \Rightarrow a \lhd \downarrow U \cap \downarrow V$,
  4. $a \leq b \Rightarrow a \lhd \{b\}$,

  where $\downarrow U = \{a \in S \mid \exists b \in U(a \leq b)\}$.
- A formal topology $(S, \leq, \lhd)$ is *set-presented* if there exists a family of subsets $C(a, i)$ of S, where $i \in I(a)$ and $a \in S$, such that
$$a \lhd U \Leftrightarrow \exists i \in I(a)(C(a, i) \subseteq U).$$

Formal topology

Let $(S, \leq, \lhd)$ be a formal topology.

- A *formal point* of a formal topology $(S, \leq, \lhd)$ is a subset $\alpha \subseteq S$ such that
  1. α is inhabited,
  2. $a, b \in \alpha \Rightarrow (\downarrow a \cap \downarrow b) \between \alpha$
  3. $a \in \alpha$ and $a \lhd U \Rightarrow U \between \alpha$.

If $(S, \leq, \lhd)$ is set-presented, then the condition 3 is equivalent to

$$\forall i \in I(a)[a \in \alpha \Rightarrow C(a, i) \between \alpha].$$

Formal topology

Proposition

Assume RRS_2-uREA *or* RDC. *Then the class of formal points of a set-presented formal topology is set-generated.*

Proof.

Note that the class of formal points is the class of models of the theory:

$$
\begin{aligned}
& \{\textstyle\bigvee_{a \in S} a\} \\
\cup \quad & \{\textstyle\bigwedge\{a, b\} \to \bigvee_{c \in \downarrow a \cap \downarrow b} c \mid a, b \in S\} \\
\cup \quad & \{a \to \textstyle\bigvee_{b \in C(a,i)} b \mid i \in I(a), a \in S\}.
\end{aligned}
$$

□

Formal topology

Corollary

Assume RRS_2-uREA *or* RDC. *Then the class of minimal formal points of a set-presented formal topology is a set.*

A formal topology $(S, \leq, \lhd)$ is T_1 if $\alpha \subseteq \beta \Rightarrow \alpha = \beta$ for each formal points α and β.

Corollary

Assume RRS_2-uREA *or* RDC. *Then the class of formal points of a set-presented* T_1 *formal topology is a set.*

Formal topology

- A *continuous morphism* from a formal topology $(S, \leq, \lhd)$ into a formal topology $(S', \leq', \lhd')$ is a relation $r \subseteq S \times S'$ such that
  1. $a\ r\ b$ and $b \lhd' V \Rightarrow a \lhd r^{-1}(V)$,
  2. $a \lhd r^{-1}(S')$,
  3. $a\ r\ b$ and $a\ r\ c \Rightarrow a \lhd r^{-1}(\downarrow b \cap \downarrow c)$.
  4. $a \lhd r^{-1} b \Rightarrow a\ r\ b$,

If $(S, \leq, \lhd)$ and $(S', \leq', \lhd')$ are set-presented, then the conditions 1, 2 and 3 are respectively equivalent to

- $\forall j \in I'(b)[a\ r\ b \Rightarrow \exists i \in I(a) \forall a' \in C(a,i) \exists b' \in C'(b,j)(a'\ r\ b')]$,
- $\exists i \in I(a) \forall a' \in C(a,i) \exists b \in S'(a'\ r\ b)$,
- $a\ r\ b$ and $a\ r\ c \Rightarrow \exists i \in I(a) \forall a' \in C(a,i) \exists d \in \downarrow b \cap \downarrow c(a'\ r\ d)$.

Formal topology

Proposition

Assume RRS_2-uREA *or* RDC. *Then the class of continuous morphisms between set-presented formal topologies is set-generated.*

Formal topology

Proof.
Note that the class R of relations satisfying the condition 1, 2 and 3 is the class of models of the theory:

$$
\begin{aligned}
& \{(a,b) \to \bigvee_{i \in I(a)} \bigwedge_{a' \in C(a,i)} \bigvee_{b' \in C'(b,j)} (a', b') \\
& \quad \mid j \in I'(b), a \in S, b \in S'\} \\
\cup \quad & \{\bigvee_{i \in I(a)} \bigwedge_{a' \in C(a,i)} \bigvee_{b \in S'} (a', b) \mid a \in S\} \\
\cup \quad & \{\bigwedge\{(a,b),(a,c)\} \to \bigvee_{i \in I(a)} \bigwedge_{a' \in C(a,i)} \bigvee_{d \in \downarrow b \cap \downarrow c} (a', d) \\
& \quad \mid a \in S, b, c \in S'\},
\end{aligned}
$$

and the class of continuous morphisms is given by

$$\{\{(a,b) \mid a \lhd r^{-1} b\} \mid r \in R\}.$$

□

Basic pair (joint work with Tatsuji Kawai)

- A *basic pair* is a triple $(X, \Vdash, S)$ of sets X and S and a relation $\Vdash \subseteq X \times S$.
- A *relation pair* between basic pairs $(X, \Vdash, S)$ and $(X', \Vdash', S')$ is a pair (r, s) of relations with $r \subseteq X \times X'$ and $s \subseteq S \times S'$ such that

$$\Vdash' \circ r = s \circ \Vdash .$$

- Two relation pairs (r_1, s_1) and (r_2, s_2) between pasic pairs $(X, \Vdash, S)$ and $(X', \Vdash', S')$ are *equivalent*, denoted by $(r_1, s_1) \sim (r_2, s_2)$, if

$$\Vdash' \circ r_1 = \Vdash' \circ r_2,$$

or equivalently

$$s_1 \circ \Vdash = s_2 \circ \Vdash .$$

Basic pair (joint work with Tatsuji Kawai)

Theorem

Assume RRS_2-uREA *or* RDC. *Then coequalizers exist in the category of basic pairs.*

Basic pair (joint work with Tatsuji Kawai)

Proof.

Let (r_1, s_1) and (r_2, s_2) be relation pair between basic pairs $(X, \Vdash, S)$ and $(X', \Vdash', S')$. Then the class

$$Q = \{U \in \mathrm{Pow}(S') \mid (s_1 \circ \Vdash)^{-1}(U) = (s_2 \circ \Vdash)^{-1}(U)\}$$

is the class of the models of the theory:

$$\begin{aligned} & \{a \to \textstyle\bigwedge_{x \in (s_1 \circ \Vdash)^{-1}(a)} \bigvee_{b \in (s_2 \circ \Vdash)(x)} b \mid a \in S'\} \\ \cup \quad & \{a \to \textstyle\bigwedge_{x \in (s_2 \circ \Vdash)^{-1}(a)} \bigvee_{b \in (s_1 \circ \Vdash)(x)} b \mid a \in S'\}. \end{aligned}$$

Let G be a generating set of Q. Then $(X', \Vdash', G)$ with a relation pair $(\mathrm{id}_{X'}, \in)$ is a coequalizer for (r_1, s_1) and (r_2, s_2). □