Applied Constructive Algebra

A case study: the computation of grade filtration

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Modern Constructive Algebra - Dedicated to Henri Lombardi
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Memories

- I first got in touch with Henri in 2001.
- At that time, I was investigating questions in control theory.
- Control theorists studied the following question:

Let A be a commutative integral domain, $K = Q(A)$ its quotient field, and $p \in K$. When is there a $c \in K$ such that:

$$H(p,c) := \begin{pmatrix} \frac{1}{1-p\,c} & \frac{p}{1-p\,c} \\ \frac{c}{1-p\,c} & \frac{1}{1-p\,c} \end{pmatrix} \in A^{2\times 2}?$$

- For an algebraist, this question was not too difficult:

It is "if and only if $J = (1, p)$ is an invertible fractional ideal of A".

$$J^{-1} = A : J = (a, b), \quad a - b\,p = 1 \quad \Rightarrow \quad c = b/a.$$

- $\forall\, p \in K,\ \exists\, c \in K :\ H(p,c) \in A^{2\times 2}$ iff A is Prüfer domain!

Memories

- Browsing the literature on Prüfer domains, I found they appeared in real algebraic geometry (Nash functions on a Nash manifold).
- I naturally asked Marie-Françoise Roy for more information.
- She told me: "Yes, I know someone who can help you:

Henri Lombardi!"

- Since then, I have followed his work on constructive algebra.
- The rings A used in control theory are mainly Banach algebras.
- But a Banach algebra A of $\dim_k(A) = \infty$ cannot be noetherian.
- So I had to develop an approach to stabilization problems based on the category of finitely presented modules over a coherent ring.
- I then discovered that Henri shared the same philosophy!

Memories

• Stabilization problems are closely related to K-theory.

stable rank, Serre's theorem, Swan-Serre's theorem, Heitmann...

• In my investigations of problems coming from control theory, talking with Henri and reading his papers were of great help.

• For a different issue in control theory, Fabiańska and I studied constructive versions of the Quillen-Suslin's theorem.

$\Rightarrow$ the Maple package QuillenSuslin.

• We then learnt about Henri and Ihsen nice work in this direction.

• I studied Stafford's theorem on projective modules over $A_n(k)$. I obtained a constructive algorithm based on Henri's lecture notes.

$\Rightarrow$ the Maple package Stafford (with Robertz).

Memories

• I could easily multiply the examples where his work had a profound influence on mine.

• More recently, with Henri, Thierry and Ihsen, we have set up meetings on constructive homological algebra.

• I would like to show you some results going in this direction.

• But first, my collaborators Mohamed Barakat, Daniel Robertz, Thomas Cluzeau, Georg Regensburger and I would like to tell him:

Chapeau bas l'Artiste et bon vent!

Solving polynomial systems

- Example: Greuel, Pfister, 02:

$$\begin{cases} P_1 = x_1^3 + x_1^2\,x_2 + x_1^2\,x_3 - x_1^2 - x_1\,x_3 - x_2\,x_3 - x_3^2 + x_3, \\ P_2 = x_1^2\,x_2\,x_3 + x_1^2\,x_2 - x_2\,x_3^2 - x_2\,x_3, \\ P_3 = x_1^2\,x_2^2 - x_1^2\,x_2 - x_2^2\,x_3 + x_2\,x_3. \end{cases}$$

The solve command of Maple returns:

- Dimension 2: $\{(x_1 = x_1,\ x_2 = x_2,\ x_3 = x_1^2)\}$.
- Dimension 1: $\{(x_1 = x_1,\ x_2 = 0,\ x_3 = -x_1 + 1)\}$,
  $\{(x_1 = \pm i,\ x_2 = x_2,\ x_3 = -1)\}$.
- Dimension 0: $\{(x_1 = 1,\ x_2 = 1,\ x_3 = -1)\}$.

Equidimensional decomposition of algebraic varieties.

Solving linear PD systems in Maple

- Maple cannot integrate the following simple linear PD system!

$$x = (x_1, x_2), \quad \partial_i = \frac{\partial}{\partial x_i}, \quad \left\{ \begin{array}{l} \partial_1^2 \, (\partial_1 - \partial_2) \, y(x) = 0, \\ \partial_1 \, \partial_2 \, (\partial_1 - \partial_2) \, y(x) = 0. \end{array} \right. \qquad (\star)$$

$$z(x) = \partial_1 \, (\partial_1 - \partial_2) \, y(x) \quad \Rightarrow \quad \left\{ \begin{array}{l} \partial_1 \, z(x) = 0, \\ \partial_2 \, z(x) = 0, \end{array} \right.$$

$$\begin{array}{ll} \Rightarrow & z = \partial_1 \, (\partial_1 - \partial_2) \, y(x) = C \\ \Rightarrow & (\partial_1 - \partial_2) \, y(x) = C \, x_1 + \phi(x_2) \\ \Rightarrow & y(x) = \psi(x_1 + x_2) + \frac{1}{2} \, C \, x_1^2 + \varphi(x_2). \end{array}$$

$$(\star) \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \partial_1 \, (\partial_1 - \partial_2) \, y(x) = z(x), \\ \partial_1 \, z(x) = 0, \\ \partial_2 \, z(x) = 0. \end{array} \right.$$

Main goal: Grade filtration

- Let $D = k[x_1, \ldots, x_n]$ or $D = A\langle \partial_1, \ldots, \partial_n \rangle$ $(\partial_i\, a = a\, \partial_i + \frac{\partial a}{\partial x_i})$,

$$A = k[x_1, \ldots, x_n],\; k(x_1, \ldots, x_n),\; k[\![x_1, \ldots, x_n]\!],\; k'\{x_1, \ldots, x_n\},$$

where k is a field of char. 0 and $k' = \mathbb{R}$ or $\mathbb{C}$, and $R \in D^{q \times p}$.

$$R\,\eta = 0$$

$$\Leftrightarrow \left(\begin{array}{cccccccc} R_0 & -S_0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & R_1 & -S_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & R_2 & -S_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & R_{n-1} & -S_{n-1} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & R_n \end{array}\right) \left(\begin{array}{c} \eta \\ \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{n-1} \\ \eta_n \end{array}\right) = 0,$$

where R_i defines a linear PD system of codimension i.

Grade (purity/torsion/bidualizing) filtration

- Equidimensional decomposition of algebraic varieties (e.g., Eisenbud-Huneke-Vasconcelos 92).

- Bidualizing complexes & spectral sequences (Grothendieck, Roos 62, Hartshorne 66, Björk 79)

The spectral sequences for the corresponding bicomplexes were made constructive by Barakat 09 $\Rightarrow$ `homalg` GAP4 (Barakat 09).

- Associated cohomology (Sato, Kashiwara 70, 78) (constructive?)

- Auslander transpose (Q. 10)

$\Rightarrow$ PurityFiltration (Q. 10).

$\Rightarrow$ `AbelianSystems` package of homalg (Barakat-Q. 10).

$\Rightarrow$ improvement of the Maple pdsolve command for PD systems.

Algebraic analysis

- Let D be a noetherian domain, $R \in D^{q\times p}$.
- Let us consider the left D-homomorphism (D-linear map):

$$\begin{array}{rcl} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} \\ \lambda = (\lambda_1 \ \dots \ \lambda_q) & \longmapsto & \lambda\, R. \end{array}$$

- We introduce the finitely presented left D-module:

$$M := \mathrm{coker}_D(.R) = D^{1\times p}/\mathrm{im}_D(.R) = D^{1\times p}/(D^{1\times q}\, R).$$

- Algebraic geometry: $M = \mathbb{Q}[x,y]/(x^2+y^2-1, x-y)$:

$$D = \mathbb{Q}[x,y], \qquad D^{1\times 2} \xrightarrow{.\left(\begin{array}{c} x^2+y^2-1 \\ x-y \end{array}\right)} D \xrightarrow{\pi} M \longrightarrow 0.$$

Free resolutions

- Definition: A finite free resolution of a left D-module M is an exact sequence of the form:

$$\ldots \xrightarrow{.R_3} D^{1\times l_2} \xrightarrow{.R_2} D^{1\times l_1} \xrightarrow{.R_1} D^{1\times l_0} \xrightarrow{\pi} M \longrightarrow 0,$$

$$R_i \in D^{l_i \times l_{i-1}}, \qquad \begin{array}{ccc} D^{1\times l_i} & \xrightarrow{.R_i} & D^{1\times l_{i-1}} \\ (d_1 \ \ldots \ d_{l_i}) & \longmapsto & (d_1 \ \ldots \ d_{l_i})\, R_i. \end{array}$$

- Algorithm: Find a basis of the compatibility conditions of the inhomogeneous system $R_i\, y = u$ by eliminating y:

$$\forall\ P \in \ker_D(.R_i), \quad P\,(R_i\, y) = P\, u \ \Rightarrow\ P\, u = 0.$$

- Gröbner/Janet bases, differential algebra, Spencer's theory...

Extension modules $\mathrm{ext}_D^i(\,\cdot\,, D)$'s

- We introduce the reduced free resolution $M_\bullet$ of M by:

$$\ldots \xrightarrow{.R_3} D^{1\times l_2} \xrightarrow{.R_2} D^{1\times l_1} \xrightarrow{.R_1} D^{1\times l_0} \longrightarrow 0 \quad (\star).$$

- Applying the functor $\hom_D(\,\cdot\,, D)$ to $(\star)$, we obtain the complex:

$$\begin{array}{ccccccccc} \ldots & \xleftarrow{R_3.} & D^{l_2} & \xleftarrow{R_2.} & D^{l_1} & \xleftarrow{R_1.} & D^{l_0} & \longleftarrow 0, & (\star\star) \\ & & & & R_1\,\eta & \longleftarrow & \eta & & \\ & & R_2\,\zeta & \longleftarrow & \zeta & & & & \end{array}$$

- The defects of exactness of $(\star\star)$ are denoted by:

$$\begin{cases} \mathrm{ext}_D^0(M, D) = \hom_D(M, D) \cong \ker_D(R_1.), \\ \mathrm{ext}_D^i(M, D) \cong \ker_D(R_{i+1}.) / \mathrm{im}_D(R_i.), \quad i \geq 1. \end{cases}$$

- Theorem: The right D-modules $\mathrm{ext}_D^i(M, D)$'s depend only on M and not on the choice of the reduced free resolution $(\star)$.

Example

- $D = \mathbb{Q}[x_1, x_2]$, $R = \begin{pmatrix} x_1^2 \\ x_1\, x_2 \end{pmatrix}$, $R_2 = (x_2 \quad -x_1)$, $R' = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 2} \xrightarrow{.R} D \xrightarrow{\pi} M = D/(x_1^2, x_1\, x_2) \longrightarrow 0.$$

- The reduced free resolution $M_\bullet$ of M is the following complex:

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 2} \xrightarrow{.R} D \longrightarrow 0. \quad (\star)$$

- Applying the functor $\hom_D(\,\cdot\,, D)$ to $(\star)$, we get the complex:

$$0 \longleftarrow D \xleftarrow{R_2.} D^2 \xleftarrow{R.} D \longleftarrow 0.$$

$$\left\{ \begin{array}{l} \mathrm{ext}^0_D(M, D) = \hom_D(M, D) \cong \ker_D(R.) = 0, \\ \mathrm{ext}^1_D(M, D) \cong \ker_D(R_2.)/\mathrm{im}_D(R.) = (R'\, D)/(R\, D) \cong D/(x_1) \neq 0, \\ \mathrm{ext}^2_D(M, D) \cong D/(R_2\, D^2) = D/(x_1, x_2) \neq 0. \end{array} \right.$$

Auslander regular rings & Cohen-Macaulay

- Definition: A ring D is a Cohen-Macaulay ring if D is noetherian equipped with a dimension function $\dim_D(\cdot)$ such that:

$$\begin{aligned}\mathrm{codim}_D(M) &:= \dim_D(D) - \dim_D(M)\\ &= j(M) := \min\{i \geq 0 \mid \mathrm{ext}_D^i(M, D) \neq 0\}.\end{aligned}$$

- Example: If $D = k[x_1, \ldots, x_n]$ or $B_n(k)$, then $\dim(D) = n$.
- Example: If $D = A_n(k)$, $\hat{\mathcal{D}}_n(k)$ or $\mathcal{D}_n(k)$, then $\dim(D) = 2n$.

- Definition: A ring D is Auslander regular if D is a noetherian ring with a finite global dimension $\mathrm{gld}(D)$ and:

$$\forall\, i \in \mathbb{N}, \quad \forall\, M \quad \text{f.g.}, \quad \forall\, N \subseteq \mathrm{ext}_D^i(M, D) \;\Rightarrow\; j(N) \geq i.$$

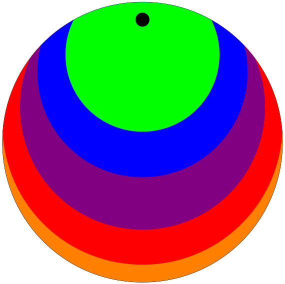
Grade filtration

- Definition: The grade filtration of M is the sequence of left D-submodules $\{M_i\}_{i=0,\ldots,n+1}$ of M defined by:

$$M_i := \{m \in M \mid j(D\,m) = \mathrm{codim}_D(D\,m) \geq i\}.$$

1. $0 = M_{n+1} \subseteq M_n \subseteq \cdots \subseteq M_2 \subseteq M_1 \subseteq M_0 = M$.
2. $L_i = M_i / M_{i-1}$ is a i-pure left D-module, i.e., $\forall \; L \subseteq L_i$:

$$j_D(L) = j_D(L_i) = i, \quad (\mathrm{codim}_D(L) = \mathrm{codim}_D(L_i) = i)).$$



Example

- Let us consider the following simple linear PD system:

$$\begin{cases} \partial_1^2 \, y(x_1, x_2) = 0, \\ \partial_1 \, \partial_2 \, y(x_1, x_2) = 0. \end{cases}$$

- The $D = \mathbb{Q}[\partial_1, \partial_2]$-module $M = D/(\partial_1^2, \partial_1 \, \partial_2)$ is defined by the generator $z = \pi(1)$ and the relations:

$$\partial_1^2 \, z = 0, \quad \partial_1 \, \partial_2 \, z = 0.$$

- $z_1 = \partial_2 \, y \in M_1 = \{m \in M \mid \mathrm{codim}_D(m) \geq 1\}$ since $\partial_1 \, z_1 = 0$:

$$\dim_D(\mathrm{ann}_D(z_1)) = \dim_D(D/(\partial_1)) = 1.$$

- $z_2 = \partial_1 \, y \in M_2 = \{m \in M \mid \mathrm{codim}_D(m) \geq 2\}$ since:

$$\begin{cases} \partial_1 \, z_2 = 0, \\ \partial_2 \, z_2 = 0, \end{cases}$$

$$\dim_D(\mathrm{ann}_D(z_2)) = \dim_D(D/(\partial_1, \partial_2)) = 0.$$

Grade filtration

- Let us consider the beginning of a finite free resolution of M:

$$0 \longleftarrow M \xleftarrow{\pi} D^{1\times p_0} \xleftarrow{.R_1} D^{1\times p_1} \xleftarrow{.R_2} D^{1\times p_2} \xleftarrow{.R_3} D^{1\times p_3} \xleftarrow{.R_4} \cdots$$

- Applying $\hom_D(\,\cdot\,, D)$ to $M_\bullet$, we get the following complex:

$$0 \longrightarrow D^{p_0} \xrightarrow{R_1 .} D^{p_1} \xrightarrow{R_2 .} D^{p_2} \xrightarrow{R_3 .} D^{p_3} \xrightarrow{R_4 .} \cdots$$

$$\begin{cases} \operatorname{ext}^1_D(M, D) = \ker_D(R_2.)/\operatorname{im}_D(R_1.), \\ \operatorname{ext}^2_D(M, D) = \ker_D(R_3.)/\operatorname{im}_D(R_2.), \\ \operatorname{ext}^3_D(M, D) = \ker_D(R_4.)/\operatorname{im}_D(R_3.), \ldots \end{cases}$$

Grade filtration

- Let use the notations: $R_{ii} = R_i$, $p_{ii} = p_i$.

$$0 \longrightarrow D^{p_{00}} \xrightarrow{R_{11}.} D^{p_{11}} \xrightarrow{R_{22}.} D^{p_{22}} \xrightarrow{R_{33}.} D^{p_{33}} \xrightarrow{R_{44}.} \ldots$$

- Let us introduce the so-called Auslander transposes:

$$N_{ii} = D_i^p/(R_i\, D^{p_{i-1}}) = D^{p_{ii}}/(R_{ii}\, D^{p_{(i-1)i}}).$$

- We get the following exact sequences ($p_{i\,i+1} = p_{ii}$):

$$\begin{array}{ccccccccccc}
 & & & & D^{p_{23}} & \xrightarrow{R_{33}.} & D^{p_{33}} & \xrightarrow{\kappa_{33}} & N_{33} & \longrightarrow & 0 \\
 & & & & \| & & & & & & \\
 & & D^{p_{12}} & \xrightarrow{R_{22}.} & D^{p_{22}} & \xrightarrow{\kappa_{22}} & N_{22} & \longrightarrow & 0 & & \\
 & & \| & & & & & & & & \\
D^{p_{01}} & \xrightarrow{R_{11}.} & D^{p_{11}} & \xrightarrow{\kappa_{11}} & N_{11} & \longrightarrow & 0 & & & &
\end{array}$$

Grade filtration

- Let us consider the beginning of the free resolution of the N_{ii}'s:

$$
\begin{array}{ccccccccccccc}
D^{p_{-13}} & \xrightarrow{R_{03}.} & D^{p_{03}} & \xrightarrow{R_{13}.} & D^{p_{13}} & \xrightarrow{R_{23}.} & D^{p_{23}} & \xrightarrow{R_{33}.} & D^{p_{33}} & \xrightarrow{\kappa_{33}} & N_{33} & \longrightarrow & 0 \\
 & & & & & & \| & & & & & & \\
D^{p_{-12}} & \xrightarrow{R_{02}.} & D^{p_{02}} & \xrightarrow{R_{12}.} & D^{p_{12}} & \xrightarrow{R_{22}.} & D^{p_{22}} & \xrightarrow{\kappa_{22}} & N_{22} & \longrightarrow & 0 & & \\
 & & & & \| & & & & & & & & \\
D^{p_{-11}} & \xrightarrow{R_{01}.} & D^{p_{01}} & \xrightarrow{R_{11}.} & D^{p_{11}} & \xrightarrow{\kappa_{11}} & N_{11} & \longrightarrow & 0 & & & &
\end{array}
$$

$$
\left\{
\begin{array}{l}
\mathrm{ext}^1_D(M, D) = \ker_D(R_{22}.)/\mathrm{im}_D(R_{11}.) = \mathrm{im}_D(R_{12}.)/\mathrm{im}_D(R_{11}.), \\
\mathrm{ext}^2_D(M, D) = \ker_D(R_{33}.)/\mathrm{im}_D(R_{22}.) = \mathrm{im}_D(R_{23}.)/\mathrm{im}_D(R_{22}.), \ \ldots
\end{array}
\right.
$$

$$
\Rightarrow \quad
\left\{
\begin{array}{l}
\exists \ F_{02} \in D^{p_{01} \times p_{02}} : \ R_{11} = F_{02} \, R_{12}, \\
\exists \ F_{13} \in D^{p_{12} \times p_{13}} : \ R_{33} = F_{13} \, R_{23}, \ \ldots
\end{array}
\right.
$$

Grade filtration

- We get the following commutative exact diagram:

$$\begin{array}{ccccccccccccc}
D^{p_{-13}} & \xrightarrow{R_{03}.} & D^{p_{03}} & \xrightarrow{R_{13}.} & D^{p_{13}} & \xrightarrow{R_{23}.} & D^{p_{23}} & \xrightarrow{R_{33}.} & D^{p_{33}} & \xrightarrow{\kappa_{33}} & N_{33} & \longrightarrow & 0 \\
\uparrow F_{-13}. & & \uparrow F_{03}. & & \uparrow F_{13}. & & \parallel & & & & & & \\
D^{p_{-12}} & \xrightarrow{R_{02}.} & D^{p_{02}} & \xrightarrow{R_{12}.} & D^{p_{12}} & \xrightarrow{R_{22}.} & D^{p_{22}} & \xrightarrow{\kappa_{22}} & N_{22} & \longrightarrow & 0 & & \\
\uparrow F_{-12}. & & \uparrow F_{02}. & & \parallel & & & & & & & & \\
D^{p_{-11}} & \xrightarrow{R_{01}.} & D^{p_{01}} & \xrightarrow{R_{11}.} & D^{p_{11}} & \xrightarrow{\kappa_{11}} & N_{11} & \longrightarrow & 0 & & & &
\end{array}$$

- Applying $\hom_D(\,\cdot\,, D)$, we get the following complex:

$$\begin{array}{ccccc}
D^{1\times p_{-13}} & \xleftarrow{.R_{03}} & D^{1\times p_{03}} & \xleftarrow{.R_{13}} & D^{1\times p_{13}} \\
\downarrow .F_{-13} & & \downarrow .F_{03} & & \downarrow .F_{13} \\
D^{1\times p_{-12}} & \xleftarrow{.R_{02}} & D^{1\times p_{02}} & \xleftarrow{.R_{12}} & D^{1\times p_{12}} \\
\downarrow .F_{-12} & & \downarrow .F_{02} & & \parallel \\
D^{1\times p_{-11}} & \xleftarrow{.R_{01}} & D^{1\times p_{01}} & \xleftarrow{.R_{11}} & D^{1\times p_{11}}.
\end{array}$$

- Theorem: $M_i = \{m \in M \mid \operatorname{codim}_D(D\, m) \geq i\} \cong \operatorname{ext}^i_D(N_{ii}, D)$.

$M_1 \cong \operatorname{ext}^1_D(N_{11}, D), \quad M_2 \cong \operatorname{ext}^2_D(N_{22}, D), \quad M_3 \cong \operatorname{ext}^3_D(N_{33}, D), \ \ldots$

Grade filtration

- Let us consider the following commutative exact diagram:

$$
\begin{array}{ccccccc}
D^{1\times p_{-13}} & \xleftarrow{.R_{03}} & D^{1\times p_{03}} & \xleftarrow{.R'_{13}} & D^{1\times p'_{13}} & \xleftarrow{.R'_{23}} & D^{1\times p'_{23}} \\
\downarrow .F_{-13} & & \downarrow .F_{03} & & \downarrow .F'_{13} & & \downarrow .F'_{23} \\
D^{1\times p_{-12}} & \xleftarrow{.R_{02}} & D^{1\times p_{02}} & \xleftarrow{.R'_{12}} & D^{1\times p'_{12}} & \xleftarrow{.R'_{22}} & D^{1\times p'_{22}} \\
\downarrow .F_{-12} & & \downarrow .F_{02} & & \downarrow .F'_{12} & & \downarrow .F'_{22} \\
D^{1\times p_{-11}} & \xleftarrow{.R_{01}} & D^{1\times p_{01}} & \xleftarrow{.R'_{11}} & D^{1\times p'_{11}} & \xleftarrow{.R'_{21}} & D^{1\times p'_{21}}.
\end{array}
$$

$$
\left\{
\begin{array}{l}
\ker_D(.R_{01}) = D^{1\times p_{11'}}\, R'_{11} \supseteq D^{1\times p_{11}}\, R_{11}, \\
\ker_D(.R_{02}) = D^{1\times p_{12'}}\, R'_{12} \supseteq D^{1\times p_{12}}\, R_{12}, \\
\ker_D(.R_{03}) = D^{1\times p_{13'}}\, R'_{13} \supseteq D^{1\times p_{13}}\, R_{13}, \ \ldots
\end{array}
\right.
$$

$\Rightarrow$ There exist matrices R''_{11}, R''_{12}, R''_{13}, ... such that:

$$R_{11} = R''_{11}\, R'_{11}, \quad R_{12} = R''_{12}\, R_{12}, \quad R_{13} = R''_{13}\, R'_{13}, \ \ldots$$

Grade filtration

- Theorem:

$$
\begin{cases}
M_1 = D^{1\times p'_{11}}/(D^{1\times p_{11}}\, R''_{11} + D^{1\times p'_{21}}\, R'_{21}), & \mathrm{cd} \geq 1, \\
M_2 \cong D^{1\times p'_{12}}/(D^{1\times p_{12}}\, R''_{12} + D^{1\times p'_{22}}\, R'_{22}), & \mathrm{cd} \geq 2, \\
M_3 \cong D^{1\times p'_{13}}/(D^{1\times p_{13}}\, R''_{13} + D^{1\times p'_{23}}\, R'_{23}), & \mathrm{cd} \geq 3, \\
M_0/M_1 \cong D^{1\times p_{01}}/(D^{1\times p'_{11}}\, R'_{11}) \cong M/t(M), & \mathrm{cd} = 0, \\
M_1/M_2 \cong D^{1\times p'_{11}}/(D^{1\times p_{11}}\, R''_{11} + D^{1\times p'_{21}}\, R'_{21} + D^{1\times p'_{12}}\, F'_{12}), & \mathrm{cd} = 1, \\
M_2/M_3 \cong D^{1\times p'_{12}}/(D^{1\times p_{12}}\, R''_{12} + D^{1\times p'_{22}}\, R'_{22} + D^{1\times p'_{13}}\, F'_{13}), & \mathrm{cd} = 2, \\
\ldots
\end{cases}
$$

A new presentation of M

- Theorem: $M = D^{1\times p}/(D^{1\times q}\,R) \cong \overline{M} = D^{1\times \overline{p}}/(D^{1\times \overline{q}}\,\overline{R})$, where:

$$\overline{R} = \begin{pmatrix} R'_{11} & -I_{p'_{11}} & 0 & 0 & 0 & 0 \\ 0 & F'_{12} & -I_{p'_{12}} & 0 & 0 & 0 \\ 0 & R''_{11} & 0 & 0 & 0 & 0 \\ 0 & R'_{21} & 0 & 0 & 0 & 0 \\ 0 & 0 & \vdots & \vdots & 0 & 0 \\ 0 & 0 & \vdots & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & F'_{1(n-1)} & -I_{p'_{1(n-1)}} \\ 0 & 0 & 0 & 0 & R''_{1(n-1)} & 0 \\ 0 & 0 & 0 & 0 & R'_{2(n-1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & R''_{1n} \\ 0 & 0 & 0 & 0 & 0 & R'_{2n} \end{pmatrix}.$$

A new presentation of M

If $\overline{\pi} : D^{1\times \overline{p}} \longrightarrow \overline{M}$ is the canonical projection onto $\overline{M}$, then:

$$\begin{array}{rcl} \varphi : M & \longrightarrow & \overline{M} \\ \pi(\lambda) & \longmapsto & \overline{\pi}(\lambda\,(I_{p_{01}} \quad 0 \quad \cdots \quad 0)), \end{array}$$

$$\begin{array}{rcl} \varphi^{-1} : \overline{M} & \longrightarrow & M \\ \overline{\pi}(\mu) & \longmapsto & \pi\left(\mu\left(\begin{array}{c} I_{p_{01}} \\ R'_{11} \\ F'_{12}\,R'_{11} \\ \vdots \\ F'_{1n}\cdots F'_{12}\,R'_{11} \end{array}\right)\right). \end{array}$$

Example

- Example: Let $D = \mathbb{Q}[\partial_1, \partial_2, \partial_3]$, $\partial_i = \frac{\partial}{\partial x_i}$,

$$R = \begin{pmatrix} 0 & -2\,\partial_1 & \partial_3 - 2\,\partial_2 - \partial_1 & -1 \\ 0 & \partial_3 - 2\,\partial_1 & 2\,\partial_2 - 3\,\partial_1 & 1 \\ \partial_3 & -6\,\partial_1 & -2\,\partial_2 - 5\,\partial_1 & -1 \\ 0 & \partial_2 - \partial_1 & \partial_2 - \partial_1 & 0 \\ \partial_2 & -\partial_1 & -\partial_2 - \partial_1 & 0 \\ \partial_1 & -\partial_1 & -2\,\partial_1 & 0 \end{pmatrix},$$

and the D-module $M = D^{1\times 4}/(D^{1\times 6}\,R)$ (Janet).

- Computing the purity filtration, we get $M \cong D^{1\times 11}/(D^{1\times 23}\,P)$.

Example

$$
P = \begin{pmatrix}
1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \partial_1 - 2\,\partial_2 + \partial_3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2\,\partial_1 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -2\,\partial_1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2\,\partial_1 + \partial_3 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \partial_3 & -6\,\partial_1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\partial_1 + \partial_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \partial_2 & -\partial_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \partial_1 & -\partial_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4\,\partial_1 - \partial_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4\,\partial_1 - \partial_3 & \partial_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_1 - \partial_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_1 - \partial_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\partial_3
\end{pmatrix}.
$$

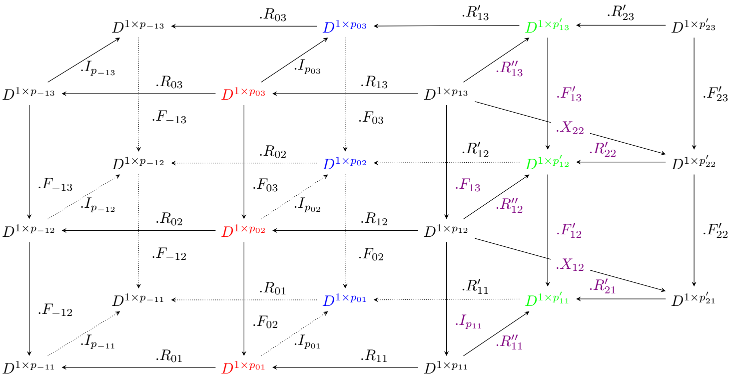
Example

$$\left\{\begin{array}{l} -2\,\partial_1\,\eta_2+\partial_3\,\eta_3-2\,\partial_2\,\eta_3-\partial_1\,\eta_3-\eta_4=0,\\ \partial_3\,\eta_2-2\,\partial_1\,\eta_2+2\,\partial_2\,\eta_3-3\,\partial_1\,\eta_3+\eta_4=0,\\ \partial_3\,\eta_1-6\,\partial_1\,\eta_2-2\,\partial_2\,\eta_3-5\,\partial_1\,\eta_3-\eta_4=0,\\ \partial_2\,\eta_2-\partial_1\,\eta_2+\partial_2\,\eta_3-\partial_1\,\eta_3=0,\\ \partial_2\,\eta_1-\partial_1\,\eta_2-\partial_2\,\eta_3-\partial_1\,\eta_3=0,\\ \partial_1\,\eta_1-\partial_1\,\eta_2-2\,\partial_1\,\eta_3=0, \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l} \eta_1(x)=\xi(x)-f_1(x_3+\frac{1}{4}\,(x_1+x_2))+c_1,\\ \eta_2(x)=-\xi(x)-f_1(x_3+\frac{1}{4}\,(x_1+x_2)),\\ \eta_3(x)=\xi(x),\\ \eta_4(x)=(\partial_1-2\,\partial_2+\partial_3)\,\xi(x)+\frac{1}{2}\,\dot{f}_1(x_3+\frac{1}{4}\,(x_1+x_2)), \end{array}\right.$$

where $x=(x_1,x_2,x_3)$ and ξ (resp., f_1, c_1) is an arbitrary function of $C^\infty(\mathbb{R}^3)$ (resp., $C^\infty(\mathbb{R})$, constant).

Purity filtration



Triangularization of linear systems