Minimal zero-dimensional extensions

Fred Richman

Florida Atlantic University

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Besançon

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The ring $\mathbf{Q} \times \mathbf{Z}_n$ is a minimal zero-dimensional extension of $\mathbf{Z}$. The set $\mathcal{P}$ consists of the primary ideals in the decomposition of the ideal (n).

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In a subsequent paper, Marcela and I weakened the one-dimensionality condition to read that every nonminimal prime ideal be contained in only finitely many prime ideals (as opposed to every nonminimal prime ideal being maximal).

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No conditions are imposed on the minimal zero-dimensional extension S of R. So, for example, the theory covers the extension $\mathbf{Q} \times \mathbf{Z}/I$ where I is an arbitrary nonzero ideal of $\mathbf{Z}$.

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- It's not closed under arbitrary intersection (it's a basis for closed sets).

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We want to form some sort of direct limit of the rings $R/I_{\mathfrak{a}}$.

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The surprise is that this condition is equivalent to $\dim R \leq 1$.

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If 0 is primary, and the nilradical of R is detachable, then $\dim R \leq 1$ if and only if $\dim R/Rx = 0$ whenever $x \in R$ is not nilpotent.
The condition $\dim R \leq 0$ is equivalent to the well-known classical arithmetic condition that for each $x \in R$, there exists n such that $Rx^n = Rx^{n+1}$.

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If k is factorial, say $k = \mathbf{Q}$, then such an admissible family over $k[X]$ is specified by assigning an ideal I_p in $F[X]$, containing some power of p, to each monic irreducible polynomial p in $k[X]$.

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Examples: Finitely generated rings of algebraic integers in a finite-dimensional extension field of $\mathbf{Q}$. Specifically, $\mathbf{Z}[2i]$.

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Yay!