QUANTUM COMPUTING LAND with CRISTALLOGRAPHIC GROUPS

by Michel Planat

All the distinctive features of quantum mechanics versus its classical counterpart crucially depends on the non-commutativity of observables. For instance, the existence of disjoint sets of mutually commuting operators implies quantum complementarity, in the sense that precise measurements on one set implies that possible outcomes of measurements on the other set are equally probable. The maximal number of such mutually disjoint sets (also called mutually unbiased bases) and their structure relates on concepts of finite projective geometry [1] and group theory, that we developed in the past five years [2]. Among such mutually commuting sets, those sharing a base of entangled states are at the origin of paradoxes named Kochen-Specker or Bell's theorems, that recognize the contradiction between the algebraic structure of eigenvalues/measurements and the corresponding expressions for the eigenstates [3]. We investigated the *zoology* and geometry of all mutually unbiased bases for spaces of prime dimension [4] or composite dimension [5].

Inhabitants in quantum computing world are qubits and tensor products of them, their houses are the mutually commuting sets, finite geometries are the villages and towns. I discovered a large city with 696 729 600 apartments that is, in mathematics, *isomorphic* to the largest crystallographic group [6], named $W(E_8)$. Groups of three inhabitants/states in the villages are in general highly connected/entangled, their contract is of the type GHZ (true tripartite union), W (three bipartite unions), CPT (a compromise), chain state (a pair can get a divorce) and so on. The population of the Pauli group P_n is 4^{n+1}, meaning that at most n people are connected (by a tensor product). The whole *unitary* world is infinite, but people usually travel in a finite part of it named the Clifford group C_n. The Clifford group divides into males/females and thus into two dipolar groups denoted $C_n^{\pm}$. Transsexual people do exist in C_n, e.g. for $n = 2$, $|P_2| = 64$, $|C_2| = 92160$ and $C_2^+ \cap C_2^- = 8$ (as described in [7].). The group C_3^+ is the largest maximal subgroup of $W(E_8)$ and indeed the aforementioned contracts are work contracts.

Following Mermin's great intuition about quantum paradoxes, I have discovered two real two-qubit matrices controling every action in C_n, the first is a braiding matrix R (of a topological character) and the second is a CPT matrix S (related to charge conjugation C, parity P and time reversal T). The dipolar group of a male type $\mathcal{C}_n^+$ is labeled by S and the dipolar group $\mathcal{C}_n^-$ of female type is labeled by R. The geometry of the octahedron is associated to the group $\langle R, S\rangle$ generated by R and S, as already anticipated in the XIXth century [8] and thoroughly developed in the context of self-dual codes [9]. By the way, the CPT invariance is expressed, up to an isomorphism, by the single qubit Pauli group $\mathcal{P}_1$ [10] and is the kernel of the three-qubit representation of $W(E_8)$, the full Dirac group of gamma matrices is the two-qubit Pauli group $\mathcal{P}_2$ and is generated in $W(E_8)$ by matrices sustaining quantum states both of the GHZ and CPT type. Further type of three-qubit entangled states are carried by larger subgroups such as $W(F_4)$, $W(H_4)$, $W(E_6)$ and $W(E_7)$ yet to be investigated in detail, in relation to the quaternionic and octonionic representations.

All these topics, and related ones, will be introduced in my series of lectures at ZIF, on August 12 to 14, 2009.

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[1] Levay P, Saniga M and Vrana P 2008 *Phys. Rev. D* **78** 124022.
[2] Planat M and Saniga M 2008 *Quant. Inf. Comp.* **8** 127.
[3] Mermin N D 1993 *Rev. Mod. Phys.* **65** 803.
[4] Planat M and Rosu H 2005 *Eur. Phys. J. D* **36** 133.
[5] Planat M, Baboin A C and Saniga M 2008 *Int. J. Theor. Phys.* **47** 1127.
[6] Planat M 2009 Preprint 0904.3691 (quant-ph).
[7] Planat M and Solé P 2008 *J. Phys. A: Math. Theor.* **42**
[8] Klein F 1956 *Lectures on the icosahedron and the solution of equations of the fifth degree* (Dover: New York).
[9] Nebe G, Rains E M and Sloane N J A *Self-dual codes and invariant theory* (Berlin: Springer).
[10] Socolovsky M 2004 *Int. J. Theor. Phys.* **43** 1941.