Beyond usual and emergent rationalities
From pre-Newtonian to post-Einsteinian dynamics.

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SUMMARY.

The laws of dynamics are derived in a general framework – extended beyond absolute time or space-time – based on the principle of relativity, related to a Leibnizian principle of order (accounting simultaneously for a multiplicity of points of view on motion). These two principles operate directly on dynamics (not kinematics). The account for a multiplicity of co-existing points of view on motion requires the introduction of an inclusive logical framework leading to a unified hierarchical treelike structure. This formulation differs radically from the usual rational methods (Lagrange-Hamilton formalism, geometrical and group theoretical methods etc.). All these methods turn out to be only analytical models which are in fact embedded into a higher rational theory (geometrically visualized as branches of a tree: one trunk prolonged by different regular curved branches). From a physical standpoint, the generality of this multi-rational formulation allows revisiting some pre-Newtonian dynamical frameworks (Descartes and Huygens) and leads to a general solution that unifies recent dynamical models dealt with separately (doubly or deformed special relativity, space-anisotropy etc.). From a conceptual standpoint, it is shown that the development of such a Leibnizian approach of nature lies beyond the Kantian paradigm, adopted explicitly or implicitly by the physical community, at least in so far as the science of dynamics is concerned.

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ABSTRACT

The present work deals with the **principle of dynamical relativity** initiated by Huygens (parabolic world), extended here following a line of thought developed by Leibniz, leading to a distinction between two levels: **objective** possible worlds (essences) and **subjective** points of view on each world (modalities of existence). These two levels are essential to the introduction of **qualitative** and **quantitative** features, aiming at the construction of a dynamical framework allying **explanation** (intelligibility) with **exploration** (efficiency) in a unified manner avoiding thus the current "epistemological disorder" attached to this field of knowledge. Leibniz's philosophy conciliates the Aristotelian concept of substance (essence) subject to various manifestations (modalities of existence) with the principle of relativity (unknown in antiquity). This extended context will show the relevance of the Leibnizian concepts that allow the constitution of a theory including the different presently available rational models (multi-rationality): among them the two main rationalities, based on the velocity (**usual rationality:** Lagrange-Hamilton formalism) and on the rapidity (**emergent rationality:** group theory).

Such a dynamical approach requires to account for two weak assumptions: (i) replacement of the constraint corresponding to the choice of only one point of view on motion (velocity or rapidity) by a weaker constraint apt to include a multiplicity of co-existent points of view – leading to the introduction of an **inclusive logical framework –** and (ii) consideration of a general basic substrate weaker than the analytic continuum, and associated with the most general conditions imposed by the **conservation properties** on the principle of dynamical relativity (in its multi-rational Leibnizian version not to confuse with the usual space-time Galilean or Lorentzian formulations relative to the velocity which is but one point of view among others).

The first weakening amounts to pass from the quantitative realm to a qualitative one, getting a potentially unlimited number of degrees of freedom (or points of view on motion). The second weakening allows going beyond the too narrow framework of usual continuous, geometrical and analytical approaches of motion (Euclidean or hyperbolic isotropic spaces). In particular, this approach leads to the inclusion of the "**physical admissibility**" (attached to space-time methodologies such as the Lagrange-Hamilton formalism) into a wider "**dynamical admissibility**" (intimately related to the present weak formulation), apt to provide rational information impossible to grasp by the usual analytical methods. In addition to its interest for both explanation and exploration, this weak formulation shows, surprisingly, that, while it is "physically inadmissible", Descartes dynamics turns out to be "dynamically admissible".

Leibniz's assertion (partially inherited from Aristotle's ideas on substance) according to which motion is not to be restricted to a simple transport in space (locomotion) but has to be accounted for through an infinitely ordered multiplicity of points of view is a valid and pertinent statement. The achievement of such a project through the use of these weak assumptions leads to a **theory** of motion and not simply to a **model** as usually done: each point of view is associated with a particular method, while here an infinite number of **different points of view** will turn out to be grasped through a **single method**. This method develops specific tools to mix qualitative considerations (for intelligibility) and quantitative ones (for predictability). This violently

contrasts with all previous models which start by quantifying motion through **simple** properties (ratio for the velocity, additive composition law for rapidity etc.).

The distinction between "**objective"** (conserved quantities) and "**subjective"** entities (motion parameters) leads naturally to a possible definition of objectivity via "trans-subjectivity" and/or "inter-subjectivity". The trans-subjective procedure corresponds to the fact that the sole **qualitative** co-existence of different points of view (and **not a simple** quantified property associated with motion) is sufficient to deduce the **quantitative** equation of dynamics that links the two conserved quantities (energy and impulse). Thus, objectivity, via trans-subjectivity, lies beyond any simplicity criterion adopted (implicitly or explicitly) in the different models. This simplicity criterion may be illustrated by the additive character singled out in the definition of the rapidity used to develop the recent "emergent rationality" through group theoretical methods.

The quantification of motion occurs at a later step through the introduction of a **principle of order** (Leibniz's plenitude principle), mathematically expressed by a recurrent sequence allowing to generate an infinite number of properties (the velocity and rapidity emerge as two simple remarkable properties among others). More precisely, after having deduced the infinite number of solutions, one discovers that only four of them correspond to basic remarkable properties the others being more or less complicated combinations of the four basic ones (well-adapted to experimental measurements, theoretical modelling and conceptual interpretations).

This radical conceptual change is the price to pay to enter into the Leibnizian multi-rational paradigm that was short-circuited by the empirical approaches and their associated rational formulations, conceptually justified by Kant's a priori forms of intuition. Each different mathematical tool embedding one way of measuring motion develops a method based on one pre-determined specific intuition: incompatible with the Leibnizian **principle of sufficient reason**. The Leibnizian project consists in minimizing the role of intuition – associated with a priori propositions such as the consideration of space over time or any other mode of measurement – in favour of a principle of order (expressed here by a scale law), apt to encompass various ways of measuring motion. Here lies the **relational** character of the Leibnizian philosophy decreed by Kant (followed by the majority of physicists) as being unreachable by rational thought, and hence unscientific. The **absence of a proof** as to the possibility of a Leibnizian ordered multi-rationality has been abusively considered as a **proof of an absence** of such multi-rationality.

The correction of this dogmatic creed (which says more than it knows) opens the door for new explanations and further explorations. In particular, the present methodology casts some light on the Lagrange-Hamilton formalism which emerges from this weak formulation. The Lagrange-Hamilton formalism corresponds to the level that carries the unique property associated with a closed curve apt to be singled out through the least action principle. The Leibnizian multi-rational formulation exhibits the level occupied by the couple (velocity – Lagrangian) and the remarkable properties that make of it a genuine model for physical investigations. Deducing from a **higher principle,** what is usually **postulated** without a sufficient reason leads to a deeper understanding. One is also led to different explorations that go **beyond Einstein's dynamics** thanks to the generality of the basic postulates. These postulates are not – as usually done in physics – empirically posed before being rationally justified (and possibly reformulated) by some model, but they are **founded** on the strict **necessary conditions** without which the dynamical structure looses its very existence. According to Leibniz, it is by digging deep to grasp the roots of **necessity** that one obtains a **maximal number of degrees of freedom** compatible with such a necessity.

This work concerns physics as well as philosophy and history of science. The account for the principle of relativity in this extended context leads to a rational link between **post-Einsteinian** dynamics (doubly or deformed special relativity, space anisotropy, etc.) and **pre-Newtonian** ones (Huygens and Descartes dynamics). In particular, the regularization of the non-analytical Cartesian dynamics leads to fruitful results that extend Einstein's dynamics and apply to **high energy physics**. Cartesian dynamics could be accounted for on a rational ground by getting over the two epistemological obstacles associated with the continuity hypothesis and the uniqueness of motion.

PREAMBLE

Two remarks relative to the entire work and to the scientific presentation.

(i) This work, in progress, is not yet achieved. A few copies have been addressed to some colleagues for critics and corrections. Suggestions and comments are appreciated.

(ii) Those who are not interested in history and philosophy of science can move directly to the introduction. The main scientific contribution is presented in parts one and two of this work.

A – General considerations on the question of motion from antiquity to recent times.

The problem of motion goes back to Aristotle. His qualitative approach was refuted in the 17th century (Galileo, Descartes, Newton…) through (i) **physical principles** – the most famous being the relativity requirement, unknown to Aristotle – (ii) **mathematical procedures** (differential and integral calculus) and (iii) **experimental verifications**. This "modern" paradigm, differing from Aristotle's one, is at the basis of modern science. After having followed the mechanistic philosophy of nature for a while, Leibniz realized that such an approach constitutes a **severe reduction** with regard to the wider Aristotelian framework. In this framework, motion has not to be reduced to a simple **transport in space and time** through the velocity concept, which corresponds to **only one point of view** among others. He wished to rehabilitate the Aristotelian paradigm after modifying it, so that it satisfies the relativity requirement. This Leibnizian program of research was forgotten until recently, superseded by the Newtonian view on motion (followed by Lagrange, Hamilton, Lorentz, Poincaré, Einstein and the majority of physicists).

A few decades ago, the situation has changed, with the emergence of another rational framework beside the well-known rational mechanics based on the velocity concept, initiated by Lagrange and Hamilton (and then followed by Einstein – space-time physics – before being extended to other contexts especially through Noether's theorem and Gauge theories). The recent approach is based on a quite different point of view on motion, where **dynamics possesses its proper autonomy** so that motion does not need to be defined cinematically as usual, but appears in dynamics through the notion of rapidity. This point of view turns out to be very close to what Leibniz proposed to do in the 17th century concerning the autonomy of dynamics, as shown by C. Comte. Such a point of view followed partially the Leibnizian methodology and used some of the ideas developed by Langevin and Lévy-Leblond where the concept of rapidity is "basic". More precisely, J.M. Lévy-Leblond and C. Comte showed that **the rapidity parameter is physical** and not simply a mathematical trick, introduced to simplify calculations, as usually believed. Today an increasing number of scientists recognize the existence of this "emergent rationality" that complements the "usual one", and a number of works are devoted to this subject matter, especially in high energy physics (where the velocity concept looses its operational character because of its asymptotical behaviour). However, in spite of the closeness of what we call the "emergent rationality" to some of the Leibnizian ideas, this rational framework is, like the conventional one, an insufficient analytical method: one starts by defining motion through only one point of view, and then performs all the construction on such a definition.

Such a purely analytical way of thinking is not fully Leibnizian, since Leibniz's natural philosophy aimed at constructing a framework apt to include an infinite multiplicity of co-existent points of view on a given reality (here dynamics). Such a claim requires the use of a specific methodology in order to perform mathematical operations linking the different co-existent ordered points of view. The lack of such an order in conventional approaches is responsible for some structural poorness inherent to the insufficient analytical methodology (sequential reasoning), methodology by essence unable to gather the different points of view in a rational and formal manner. Such an "absence" leaves the door wide open to ambiguities and controversies, since one does not define things appropriately by including them into a unified **"relational"** framework, apt to provide a specific place for each point of view in the whole ordered structure (structure inexistent in the usual dynamical framework). In brief, one may say that the different approaches constitute only **models of motion** while Leibniz's ideal consists in the construction of **a theory of motion** including the presently available models and revealing other perspectives. What appears today in the recent formulations of dynamics was already present, but only in germ, in 17^{th} century dynamics through Descartes, Huygens and Newton different investigations on motion. Let us also recall, especially, that the question of a bounded velocity – recognized only in 20^{th} century physics – was seriously discussed by Huygens, Leibniz and others as it will be developed later on.

One should however underline that Leibniz did not provide an explicit construction of such a dynamics. He used to consider this discipline as a particular (although major) application of his general natural philosophy among different other physical and mathematical investigations. To explain his involvement in different disciplines, let us remind that Leibniz tells us that one should take into account his basic philosophy associated with the doubly infinite multiplicities associated with points of view and possible worlds. These two kinds of multiplicities (an inclusive and exclusive one) turn out to be a precious tool of investigation in dynamics as it will be shown in this work. ***As long as one interprets Leibniz's approach in the conventional analytical paradigm it seems contradictory because this paradigm is too narrow. The right way to verify the degrees of pertinence and validity of Leibniz different assertions and critics necessitates the construction of a sufficiently wide framework following his line of thought, instead of contenting oneself with more or less valid and partial articulations that lead most of the time to illegitimate analogies and fallacious conclusions.***

<u>*Detection of some misinterpretations.*</u>

Since Leibniz general interests were numerous and various, these led to different misunderstandings. Dealing with the articulation between the **"One"** and the **"many"**, some scientists attributed the origin of the "principle of least action" to Leibniz where "the least of all possible actions" was associated with "the best of all possible worlds". Another articulation associated with the priority of the "qualitative" features with respect to "quantitative" ones led some authors to attribute topology to Leibniz, arguing that topology is qualitative while geometry is quantitative. A closer look at Leibniz's methodology in the light of the present formulation (particularly what concerns trans-subjectivity) shows that, in both cases, these attributions are not convincing (and even contradictory for the first case as shown later on). Leibniz is neither the father of topology nor of the least action principle, even if he made some effort in this direction (the definition of action is effectively due to him). On the other hand and especially with the discovery of Gödel's theorems, many scholars presented as impossible, Leibniz's desire to construct what he called "a universal characteristics", a general formal language capable of resolving a number of controversies. It is shown here that such a task is possible and that Leibniz's dream was not dead, at least in dynamics, where many points are subject to controversies. The present Leibnizian formulation contributes actively to absorb many of them showing, in dynamics, the existence of an ***ontological order hidden behind the current epistemological disorder*** (due to the lack of formalisation), rendering possible the co-existence of different points of view on motion. All these misconceptions and misunderstandings (in favour of Leibniz or against him), are a direct consequence of the fact that one may articulate the "One" with the "many", the "qualitative" with the "quantitative", "order" with "disorder" or the "universal" with the "regional" in many different ways, so that one should be very cautious about the analogies one uses when dealing with a specific subject matter.

If one has to retain one essential and absolutely necessary assertion typically Leibnizian and associated with dynamics then, one has to construct a framework compatible with the following doubly infinite structure composed of an **inclusive multiplicity** and an **exclusive one**. The inclusive character will account for the different co-existent points of view, and the exclusive one will account for the possible worlds, from which only one should be retained. This allows getting a well determined approach necessary in the construction of a predictive theory. A summary of this – typically Leibnizian – construction is given by the following proposition asserting "the existence of a multiplicity of co-existent points of view on the best of all possible worlds". One remarkable point has been overlooked by the philosophers and historians of science specialized in Leibnizian dynamics as well as by the physicists: the account for points of view and possible worlds are not only associated with the internal logic of his general philosophy but it may easily be revealed inside the framework of 17th century dynamics provided one does not adopt the too restrictive Newtonian empirical paradigm (associated with Hamilton-Lagrange rationality), that retains only one point of view on motion rejecting all others. We have here a typical example of the devastation produced by hyper specialization, and of the necessity of an interdisciplinary framework to deal coherently and directly with Leibniz's methodology. In adopting one unnecessary perspective, one deprives it from its essential relational features, impossible to be revealed without its link to other "subjective" perspectives. As it will be shown later on, the essence of the Leibnizian dynamical framework lies either between these "subjective" perspectives (inter-subjectivity) or beyond any one of them (trans-subjectivity).

Specific considerations about Leibniz conciliatory approach of motion.

If Leibniz did not construct a self consistent dynamical framework comparable to Newton's (adopted by Lagrange, Hamilton and Einstein), he nevertheless addressed a number of critics and positive assertions (sufficiently precise) as to what should be a real dynamical theory, that he distinguishes from the simple dynamical models developed by Descartes, Huygens and Newton: each of which dealing with one point of view on motion. The Leibnizian idea, associated with the existence of an, a priori infinite, multiplicity of points of view on motion, that may be accounted for through a specific formal language apt to resolve a number of problems, (among which the famous "vis-viva controversy") has never been seriously considered. The account for such an infinite ordered multiplicity appears to be a metaphysical assertion. The Aristotelian concept of **substance with its various modalities of existence** that Leibniz wished to adapt to the **relativity principle** does not have any counterpart in positive science, in its application as in its principle. The present work shows that this initially metaphysical idea is, contrary to what is usually believed, compatible with the basic structure of dynamics. It only necessitates the construction of an appropriate formal language to deal with this subject matter. This approach rooted in antiquity and pursued in the middle ages and the renaissance through the belief in a possible construction of what is called a "universal characteristic" capable of dealing with dynamical natural phenomena in a completely rational way, was abandoned with the advent of the analytical, space-time, parabolic framework due to Newton (empiricism) and rendered rational by Lagrange and Hamilton. (This modern rationalism is partial as compared to Leibniz's one which is multiple and more complete as shown in this work). Twentieth century (Einsteinian) dynamics simply replaced the **parabolic framework** by a **hyperbolic one** leaving the question of **infinite multiplicity of points of view out of order**, as well as the question of continuity and analyticity. This does not only have consequences on modern physics but also on our comprehension of the first dynamical approaches. (To illustrate this point one may show that this leads to the rejection of Descartes dynamics. This irregular dynamics can only be judged through the development of appropriate methods capable of accounting for a multiplicity of points of view and dealing with a framework which is neither necessarily analytical nor continuous everywhere).

One of the main points associated with the Leibnizian methodology is to underline the priority of the conceptual qualitative physical principles with respect to the quantitative mathematical ones associated with the presently available approaches of dynamics. Favouring and selecting from the start, not only a regular continuum but also one specific quantitative idea to deal with motion (velocity for the "usual" rationality and rapidity for the "emergent" one) amounts to favouring mathematics on physics, since one admits implicitly that there is nothing to say about relativity outside the continuity hypothesis as well as outside the velocity or rapidity concepts. These are false statements as shown in this work. Firstly, the **principle of relativity** may be dealt with in a **qualitative way**, independently of any specific definition of motion. One operates only on the potential existence of such a concept, and not on any specific actual existence, as in the case of the velocity or rapidity. Indeed, since the same dynamics may be accounted for through one point of view or another, then it is logical to inquire about the **possibility of** such a **dynamics without recourse to any specific point of view**. Secondly, the principle of relativity is not only independent of any specific definition of motion, but one may deal with this principle in a framework which is **neither analytical nor continuous**. It is only when these two weakening procedures (leading to a multiplicity of points of view and getting beyond the continuity hypothesis), have been well formalized that one may obtain a rational judgment on an irregular non-analytic framework (such as the Cartesian one).

B– Three key points: analogy, intuition and efficiency of mathematics, at the basis of this long term work (1977-2007).

Initially, this work was not associated with a well-identified "research program". It is the result of some analogies, intuitions and mathematical procedures developed in parallel to my initial formation in theoretical mechanics (University of "Pierre et Marie curie", Paris VI), my later research in nonlinear, complex continuous structures (interactions between mechanical, thermo-dynamical, electro-magnetic and semi-conducting effects) in the presence of singular surfaces and interfaces and to a lesser degree in near field acoustic microscopy. After having worked for many years on these subject matters, I realized that the combination of some analogies, intuitions and mathematical properties of differential calculus collected from different disciplines, may be of some use for a new understanding and extension of basic physics. The **analogy** associated with the “linear oscillator” (elliptic structure) and Einstein’s dynamics (hyperbolic structure) on the one hand (Appendix P) and the role played by the derivative in this context on the other hand, were significant on the conceptual and structural levels, relatively to the present Leibnizian formulation. The association of Leibniz to this approach is due to different convergent coincidences that I encountered in physics and mathematics as well as in history and philosophy of science through the works of C. Comte, J. Barbour, K. Gödel, H. Reichenbach, R. Thom, A. N. Whitehead, M. Serres, Y. Belaval, M. Parmentier, L. Bouquiaux, C. Frémont and others. In particular, I discovered that the typically Leibnizian **intuition** of considering a physical point (state of rest, for example, with respect to some fixed reference frame) as an accumulation point (convergence point of a family of infinite regular curves that coincide locally : tree-like structure including one trunk, from which arises an infinite number of branches) applies to dynamics. The mathematical formulation of this intuition that led to the construction of a tree-like structure, each branch constituting one point of view on motion, has been developed (among other things), in a paper entitled : “Relativité leibnizienne : philosophie relationniste intrinsèque et caractéristique universelle multiple”. Brochure du séminaire EPIPHYMATHS (pp.67-249) Université de Franche-Comté, UFR des Sciences et Techniques, 25000 Besançon (1992). In addition to this basic intuition, this work used extensively different analogical procedures. Being essentially structural and requiring a firm physical basis as suggested to me by J. Merker, (responsible of the Epiphymaths seminar), it seemed to me important to get a better understanding of this tree-like structure where Newton’s dynamics occupies the “trunk” while Einstein’s one, extends to different branches according to the specific properties one associates with motion : velocity, rapidity, etc. In addition to the physical principle required for a deeper understanding of the above-mentioned previous work, the present one includes different other articulations with prenewtonian and post-Einsteinian dynamics. Although the main results of this work were obtained long ago, these were discovered separately and with no direct relation to the principle of dynamical relativity which constitutes the core of the present formulation.

Apart from the intuition associating the “accumulation point” with the “state of rest”, and from the “analogy” between the “linear oscillator” and “Einstein’s dynamics”, I realized about twenty years ago that there exists a unique differential equation whose integration leads to Newton’s and Einstein’s dynamics: the two dynamics differ from each other by the choice of appropriate limit conditions (Appendix N). This example which illustrates the unifying character of differential calculus shows also the **efficiency of mathematics**, through an extension to new dynamical finite frameworks. This is obtained by simply adapting the limit conditions to account for a bounded energy so that Einstein’s dynamics is recovered when this energy is cast to infinity. I recently discovered that the extended results obtained previously by an appropriate choice of the constants of integration and those derived through the analogy with the “linear oscillator” correspond to recent dynamical formulations developed through totally different mathematical methods. This led me to re-examine this subject matter on the light of the principle of dynamical relativity initially developed by Huygens and extended by Leibniz (multiplicity of points of view) before being forgotten for centuries in favour of Newton’s and Einstein’s space-time physics through the kinematical relativities of Galileo and Lorentz.

Thanks to the degrees of freedom provided by the integration constants associated with the basic differential equation (representing the principle of dynamical relativity), one discovers that the admissible dynamics may be regular (analytical and continuous) or not according to the choice of the limit conditions that determine these constants. This allows to establish a link, not only with post-Einsteinian dynamics which turn out to be particular cases of the present formulation of dynamical relativity, but also with prenewtonian dynamics such as Huygens's and Descartes' ones. Cartesian dynamics was rejected empirically because of its incompatibility with 17th century experiments and rationally because of its irregular character especially that it was badly interpreted as shown later on. In the light of the present Leibnizian formulation, Descartes' dynamics (if well-interpreted) turns out to be locally valid (far from the rest state): compatible with the present formulation of dynamical relativity principle but unreachable by 17th century experiments as well as by the usual methodologies where motion is predetermined (defined at the beginning of these formulations). To see this, one should surmount two epistemological obstacles: admitting the existence of different points of view on motion (which are not predetermined as usually done) and realizing that the principle of relativity is not to be constrained by the usual continuity hypothesis. This is rendered possible thanks to the differential form of the principle of dynamical relativity, expressed independently of any specific definition of motion. Such a differential form is not only capable to **judge** the admissibility of a given dynamics (local or global, regular or irregular) but it may also **remedy** the local character (regular or irregular) of a dynamics by simply rearranging and interpreting differently the constants of integration. In a word, it plays the two significant roles of a judge and a physician. In particular, it was a big surprise to me to discover that, the regularization of Descartes' dynamics – local and irregular – does not lead only to a global and regular structure (AppendixL) that includes Newton's dynamics (as proposed indirectly by Leibniz) but also Einstein's one. Leibniz's proposal was indirect in the sense that in realizing that any even function leads locally to Newton's dynamics, the regularization of Descartes structure ($m|u| = |p|$) which respects the parity criterion, enters in the class of even functions and hence includes automatically Newton's local dynamics.

C – Origin of dynamics: usual mathematical and present conceptual interpretations.

A specific reflection on Descartes dynamics and its interpretation by physicists and mathematicians on the one hand and by historians and philosophers of science on the other hand is instructive for at least two correlated reasons. The first is that each one of the two different communities (science and humanities) tells a different story concerning the same fact! This is a direct consequence of two kinds of conceptualizations: the physicist founds his arguments on mathematical formalisation while the historian uses a philosophical argument associated with positive active substance. More precisely, Descartes' insufficiently clear concept of motion usually expressed by mu with u positive is extended by the physicist to mu with u positive and negative (a natural mathematical extension from $u \in R^+$ to $u \in R$) leading thus to the modern definition of impulse. With such an extension suggested by mathematics, the physical story (concerning the 17th century controversy on dynamics) runs as follows: The "vis viva controversy" between the followers of Descartes (mu) and those of Huygens and Leibniz (mu^2) is a false debate since the two expressions are needed to solve the dynamical problem of elastic collisions. Thus, according to the popular physical wisdom, the "vis viva controversy" is but a "pseudo-debate"; a sort of sterile historical accident with no particular interest to physics. This story that favours the idea of progress in knowledge is satisfying to the mind, but it is unfortunately false from a historical and philosophical standpoints. The situation is totally different for the historian or philosopher of science who considers that the Cartesian form mu should be replaced by its modulus $m|u|$ in order to respect the positive character of the Cartesian

conceptual framework (active substance). With this interpretation the debate becomes a true one and the problem may be tackled with, scientifically, provided one distinguishes between the following two couples of equations: (mu, $m|u|$) for Descartes and (mu, mu^2) for Huygens-Leibniz. When the 18th century physicists who followed d'Alembert rejected Descartes' system and considered that the systems of Huygens and Newton (mu, $1/2mu^2 + V$) were valid and equivalent in so far as conservation properties are concerned, they had in mind the above three different couples.
It is important never to forget that what is meant here by Descartes' dynamics is the following system of equations: (mu, $m|u|$). This structure is the one adopted by early rational scientists wishing to be as close as possible to Descartes' ill-posed and partial formulation. If one does not adapt Descartes' system to such a logical dynamical requirement, then, no possible comparison with other approaches can be made on a rational ground. Another crucial point is that the meaning of u differs from one system to another as it will be justified later on through the Leibnizian idea of multiplicity of points of view on motion where the velocity concept, at the basis of space-time formulations (rationally defined by the Lagrange-Hamilton formalism) is one point of view among others. This difference requires the introduction of formal distinctions such as u, v and w (defined later on) or more generally v_μ: the Greek index representing the multiplicity of points of view (a priori, unlimited) governed by a recurrent series of functions as shown in this work.

Danger of purely mathematical extensions and importance of philosophical considerations.
Apart from the physical (conservation properties and relativity principle) and mathematical aspects (potentialities of integro-differential equations), this work presents also conceptual or philosophical features, at least, at three levels. The first of these has just been evoked above since the present work starts with the more adequate and complete conceptualization close to the proposal of historians and philosophers and not with the truncated one usually presented by physicists and mathematicians [passage from mu with $u \in R^+$ to $u \in R$ that violates the positive character of Cartesian dynamics. According to Descartes, it is meaningless to evoke an active substance which is less than nothing (negative values)]. We have here, a significant dynamical example at the basis of physics, showing the **danger of purely mathematical extensions that do not account for the underlying concepts.** The second philosophical feature is due to the fact that unlike usual physical formulations which start by defining motion in a specific quantitative way (ratio of space over time for velocity and additive property for rapidity) to which one associates a measurement process, the application of the Leibnizian idea of infinite multiplicity of points of view cannot be said to be physical in a conventional sense. It is neither compatible with the quantitative character of physics nor with the correlated measurement process since it is impossible to actualize an infinite number of ways by which motion can be measured. This philosophical feature is attached to the very origin of metaphysics and philosophy dealing with the problem of the "One" and the "many" in a well-articulated non-contradictory manner. As usually claimed: **philosophy** anchors its methodology in the **infinite** while **physics** is rooted in the **finite.** In the present Leibnizian formulation, the existence of motion is postulated in a **qualitative** way, independently of any specific modality of measurement. This distinction between what is potentially given and what is actually realized and associated with a specific measurement will play a major role in the present work. According to Leibniz, it is the price to be paid to get a real theory of motion including an infinite multiplicity of points of view in a unified

framework, and not only simple models each based on a unique point of view and using a different methodology (variation principles for the velocity concept and group theoretical methods for the rapidity). Strictly speaking, the present methodology is somehow metaphysical (as advocated by Leibniz), in the sense that one of its basic postulates remains qualitative operating simultaneously with an infinite number of potential points of view lying beyond usual physical methodologies that select one and only one point of view on motion to which a specific formalism is associated. At a later stage, when the different points of view are actualized and specified, one operates with one of them in practice while the others exist only potentially. These are ready to be actualized when the point of view dealt with reaches the limit of its validity or when the properties required by the situation under study are not quite appropriate. Leibniz was very clear on this when he writes that the properties associated with one point of view are exhibited at the expense of numerous other remarkable properties revealed progressively through the unfolding of the different other points of view. On shedding light on one thing one casts the rest into darkness. These degrees of freedom provided by a formulation of a Leibnizian type, weakens the rigidity of the usual models each of which showing one and only one facet of dynamics. In addition to its malleability and practical efficiency, this multifaceted formulation apt to tackle with a problem from different angles or perspectives contribute to a better intelligibility.

Quality versus quantity.

Unlike the presently available methods that reach the core of dynamics (quantitative relation between conserved entities) after having adopted a specified quantitative point of view, the Leibnizian formulation may reach this same quantitative core without any particular specification about motion. Only the potential qualitative existence of unlimited and undetermined points of view is sufficient to deduce the quantitative fundamental equation of dynamics. This passage from a qualitative realm (expressing the conserved entities with respect to motion through an undetermined form) to a quantitative one (expressing the relation between conserved entities in a well determined way), specific to Leibniz's methodology and absent from the usual ones, allows to determine the compatibility of a given dynamics with the principle of dynamical relativity. The lack of such a qualitative methodology constitutes the main reason for which Descartes dynamics was wrongly judged to be false from empirical as well as rational standpoints. More precisely, if one adopts Huygens-Leibniz rationality (mw, $m|w|$) or Lagrange-Hamilton one (mv, $m|v|$) where the w and v represent the two different points of view developed explicitly later on in this work, then one discovers that Descartes dynamics is to be rejected. Such a rejection is illegitimate because the application of predetermined points of view only shows that these are not appropriate. But this does not mean that Descartes dynamics is intrinsically false. To be entirely false, one should show that whatever the adopted point of view, Descartes dynamics does not verify the constraint imposed by the principle of dynamical relativity. In a Leibnizian methodology, where the points of view are not predetermined, one is able to show if among the infinite multiplicity of points of view one of these is compatible with the principle of dynamical relativity or not. This methodology applies to any given dynamics and shows that very few are admissible. Thus, in spite of the infinite multiplicity of degrees of freedom that it admits, (corresponding to internal parameters), this method remains very restrictive which is an essential requirement without which no predictive science is possible. The attention has been focused here on Descartes dynamics (or more precisely what d'Alembert or any rational scientist would associate with Descartes ill-posed and partial dynamics), for two different reasons, only one of them has been evoked here. This reason is that contrary to what is believed since d'Alembert, the above-mentioned Descartes dynamics is not to be rejected but only corrected (as advocated by Leibniz). One should firstly determine the point of view appropriate to this dynamics and secondly specify the scale at which this dynamics is valid. This last specification will lead us to the second reason of our interest in Descartes' dynamics. This reason is intimately related to its irregular character that requires a methodology capable to deal with non-analytic functions. This is dealt with in Appendix C. Here, we firstly emphasize the importance of the generation of a possible fundamental quantitative equation of dynamics relating the two conserved entities without having to specify any quantitative point of view keeping the door open for an infinite number of degrees of freedom ordered at a later stage. Then, we show how this operates on Descartes' dynamics without entering into details (provided in the main text and in the Appendices). In order to avoid any ambiguity, let us finally emphasize that although Descartes' dynamics as such, suffers various discrepancies and crude errors as shown by many historians of science, its natural completeness in order to get a well-posed dynamical problem (by the early rational scientists like Huygens, Leibniz, d'Alembert and possibly Lagrange) is not devoid of interest as is usually believed. It is even shown that from a structural point of view, the irregular and local Cartesian dynamics includes more potentialities than the regular and local dynamics of Huygens or Newton (mathematically equivalent but conceptually different as it will be shown). This is due to the fact that these last local dynamical frameworks are degenerate (all the points of view are fused together at this specific scale where only the trunk of the tree-like structure is revealed) while Descartes dynamics is local and irregular but not degenerate. These considerations cannot be tackled with in the usual models of modern dynamics because of the rigidity of each of them based on a predetermined point of view and exhibiting one specific branch, unable to reach the unity (tree-like structure) as well as the diversity (infinite number of degrees of freedom each associated geometrically to one branch) provided by the Leibnizian methodology.

D – Two important remarks concerning Descartes dynamics and the Leibnizian idea of multiplicity of points of view on motion.

The aim of these remarks is to avoid some ambiguities. Descartes is not usually recognized as an author who contributed actively to the science of mechanics. Yet, he favoured two main qualitative mechanical ideas associated with conservation laws and inertial motion and one quantitative dynamics ($Q = m\,|u|$). It seems important to specify the facet of Descartes physics we are interested in. It is also important to explain what is meant by multiplicity of points of view in the present work as compared to the same appellation in the mechanical community.

Descartes dynamics and its relation to conventional dynamics.

Let us recall that Cartesian dynamics is proportional to $|u|$ by opposition to Huygens or Newton's dynamics which are proportional to u^2. It is shown that, as long as one interprets the variable u as the velocity concept or as the more recent one called rapidity, the Cartesian structure has to be rejected. These two predetermined points of view on motion seem not to be consistent with the relativity requirement in the Cartesian case. However, if one associates some undetermined form using the degrees of freedom provided by the Leibnizian methodology, then one shows that among the few admissible dynamics, surprisingly Descartes dynamics turns out to be admissible since compatible with the relativity principle in spite of its irregular character. To see that the

prenewtonian dynamical approach proposed by Descartes is compatible with the relativity requirement, one should be able to deal with the relativity principle independently of the analytical framework and without recourse to any of the “usual” or “emergent” rationalities in so far as the interpretation of motion is concerned. This explains why none of the two presently available rationalities is able to deal coherently with a dynamical framework of the Cartesian type. Cartesian dynamics cumulates two difficulties requiring the construction of a doubly generalized framework that goes beyond the presently available methods, firstly because of the lack of analyticity and secondly, because of the lack of a qualitative procedure capable to detect the well-adapted point of view associated with such a dynamical framework.
It should be emphasized that we are not defending Descartes comprehension of dynamics (full of practical and theoretical errors, some of which were corrected initially by Huygens and Leibniz). However, in spite of these errors, it turns out that all what was proposed by Descartes has not to be rejected and the question of the absolute value of impulse deserves a special attention since it underlines the limit of validity of our continuous approach of dynamics. One should distinguish different facets of Descartes investigations without falling in the trap of those who denigrate him and those who consider him as a genius.

<u>*Points of view in the usual mechanical framework compared to the present approach.*</u>
In order to avoid any ambiguity one should recall that in classical mechanics one distinguishes between three different methodologies: (i) the Newtonian vectorial approach where the concept of force is primary, (ii) the d’Alembertian scalar approach also called “principle of virtual work” where the energy concept is privileged and (iii) the Lagrangian approach also named “principle of least action” initiated by Maupertuis and developed later on by Lagrange and Hamilton. The two last principles are not always distinguished by physicists, since both approaches are based on a scalar quantity having the dimension of energy. However, when dealing with complex media including dissipative phenomena and irreversible processes, the Lagrange-Hamilton formalism reaches the limits of its validity, since it applies only to “forces” that derive from potentials, which is not the case of the principle of virtual work. This is well-known in the mechanical community, where a net distinction is made between the two different scalar principles. It should be emphasized that, in spite of the net distinction between these three approaches, all of them deal with the **velocity concept**. Thus, the “multiplicity of points of view” evoked in the mechanical community, does not correspond to what is here called a “multiplicity of points of view”, since the three different methods are imbedded in conventional space-time physics, where motion is accounted for exclusively through the velocity concept.

In reason of the interdisciplinary character of this work, one may distinguish in the bibliography, three types of references: historical and philosophical ones [1-6], specialized recent ones [7,8], mathematical and dynamical ones [9-16] and references where some needed conceptual elements are underlined [17-43].

INTRODUCTION

Usual rationality.

According to Leibniz [1-6], each time one achieves a tiny section of a wide framework and links it to some local natural phenomena, the joy of the discovery diverts the attention leaving the major part unachieved. That is what happened effectively in classical dynamics where one meets three kinds of constraints associated with (i) continuity, (ii) velocity and (iii) parabolicity and/or hyperbolicity. ***(i) Continuity:*** This first constraint is associated with the fact that "rational mechanics" takes place in a continuous and analytical framework. ***(ii) Velocity:*** motion is described cinematically through a simple ratio between space and time. ***(iii) Parabolicity and hyperbolicity:*** Last but not least, Newtonian dynamics is confined dynamically into a parabolic framework. With the advent of Einstein's dynamics, only the last confinement has been relaxed passing from a parabolic world to a hyperbolic one. The main principle at the basis of these two constructions is the principle of kinematical relativity expressed through two different transformations due to Galileo and Lorentz, expressed through the so-called Lagrange-Hamilton formalism. These transformations remain at the basis of our dynamical approaches in spite of some recent developments [7-8].

Emergent rationality.

In opposition to the line of thought developed by "space-time physics" whose rationality is given by Lagrange-Hamilton formalism, one encounters a new rational procedure based on "group theory" that we call "emergent rationality". In this new rational framework, motion is directly accounted for in a dynamical manner. The recourse to kinematics does not constitute an absolute necessity anymore, as shown in numerous recent practical, theoretical and epistemological works. The notion of velocity is replaced by rapidity. In this regard, it is worth noting that this "emergent rationality" uses the same procedure as the one developed by Huygens in the 17^{th} century, except that the latter has been applied in a particular case (see Ref.[1]) . The notion of rapidity was firstly considered as a purely mathematical trick that facilitates some calculations before being elevated to the rank of a real physical entity (as shown by many authors and particularly by J.M. Lévy-Leblond and C. Comte [13-16] who showed the possible autonomy of dynamics).

Multiple rationality.

Before the advent of empirical science with Newton and his followers, the rational tradition initiated by Descartes was predominant, minimizing any recourse to experiment in the construction of a dynamical framework. In this tradition, Leibniz's ideal [2-6] was to construct a totally rational dynamical framework, based on two qualitative requirements originated in the Aristotelian concept of active substance and in the Galilean relativity principle.

Aristotelian active substance: Aristotle's substance and its activity through motion should not be reduced to the Newtonian concept of mass (inert entity) and to the velocity (transport in space). Leibniz associates Aristotle's active substance (an objective being) and its various modalities of existence (subjective beings), with conservation principles, expressed according to an infinite number of different perspectives (each of them constituting one point of view on motion). To be mathematically consistent, such a wide framework requires going beyond conventional analytical methods, so that one may include explicitly the Aristotelico-Leibnizian perspectives: the different points of view should be correlated to each other, through an additional discrete formalism, on which one may operate rationally. Semantically, one should distinguish between two classes of variable entities: objective concepts, associated with conservation properties such as energy and impulse, and subjective ones relative to motion, such as the notions of velocity and rapidity, as well as a number of other points of view that will be determined and specified.

Galilean relativity principle: Leibniz follows the line of thought developed by Huygens who made a positive use of the principle of Galilean relativity, through which impulse is derived from energy. Rediscovered recently and used in different papers [11, 16], this procedure is dealt with in an analytical way, that turns out to be more particular than the Leibnizian requirements. These are associated with the various modalities of existence, explicitly accounted for through the above-mentioned specific discrete formalism. This formalism leads to an infinite number of points of view, governed by a recurrent series of functions. Four of them are basic, singular and susceptible of a physical interpretation, the others being more or less complicated combinations of these four basic ones.

Some remarkable consequences (necessity of an inclusive logical framework).

This radical change, carried out by this Leibnizian methodology, shows that what is usually thought of as being objective such as the velocity, should be looked at more subtly. If the **measurement** of the velocity (usual rationality) or rapidity (emergent rationality) is effectively objective, their existence is subjective since they constitute different particular points of view on the same reality (here dynamics). Such distinctions are not necessary in dealing with the "emergent and usual rationalities": each one constitutes one analytical **model** while the Leibnizian approach aims at constructing an analytical **theory** (or co-existent multi-analytical models) including these models simultaneously and adding a number of others in a unified framework. Such a multi-analytical framework requires the introduction of a sort of **"inclusive logic"** apt to distinguish between the different **co-existent** points of view on motion and to operate on them. The lack of such distinctions leads to possible ambiguities, paradoxes and contradictions. Thus, notions such as the velocity and the rapidity are to be associated with what one may call an "objectively measured subjective entity". This change of denomination is not trivial but it leads to a deep change of method. More precisely, **if dynamics may be looked at from different "subjective" perspectives (various modalities of existence) then one may inquire about the existence of a method of investigation capable to operate beyond any subjective point of view, a sort of "trans-subjective procedure"**. Here lies one of the main points through which the qualitative features will meet the quantitative ones. The presence of different perspectives in a unique framework leads also to correlations between the different subjective entities, defining thus what may be called "inter-subjectivity", impossible to reveal in conventional models. This deep change leading to new **epistemological distinctions** generates a change at the **ontological level** since the distinction between essence and existence (or equivalently substance and its modalities of existence), constitutes the basis of the metaphysical framework of "ontology" that goes back to Plato and Aristotle. In particular, it will be shown that contrary to what was believed for centuries, Leibniz's metaphysical assertion as to the **"existence of an infinite inclusive multiplicity of points of view on the best of all exclusive possible worlds"** makes sense henceforth in positive science, and leads to a better understanding of dynamics, provided one constructs the appropriate framework for its possibility. Indeed, the usual analytical framework is not sufficiently wide to allow a rational judgement of dynamics as a whole: only a part is clearly explained by the Lagrange-Hamilton formalism (usual rationality). Another part is explained by the emergent rationality where rapidity is basic. (In particular, none of these two rationalities possesses a decisional power to judge the prenewtonian Descartes dynamics which does not enter in the too narrow analytical framework). In addition, whenever one includes both rationalities in a framework that reveals their complementary features, the question of considering that one is better than the other becomes logically inconsistent. Other problems are solved such as the one associated with the cohesion of substance as well as the one that distinguishes "least action" from "best world" confused by different authors as shown later on.

Dimensional consideration.

Contrary to the frameworks where multidimensionality is essential, as in electromagnetism for instance, the one dimensional dynamics associated with frontal collisions, (historically at the basis of the emergence of the energy and impulse concepts) has a clear physical meaning. Since the present work deals with two different kinds of multiplicities associated with points of view and possible worlds that will be accounted for explicitly through Greek and Latin indices, we shall not add to this a third multiplicity especially that there is already much to say in this one dimensional framework.

This work is composed of three different parts. The first one is anchored in well-known physics while the second and third parts deal with less-known features some of which are related to possible extensions of Einstein's framework while others are associated with pre-Newtonian dynamics.

FIRST PART

Summary.

We firstly deal with the principle of dynamical relativity, directly associated with conservation properties. Two versions are proposed: a subjective qualitative and trans-subjective quantitative one. Secondly, an extension of Einstein's dynamics is proposed, including an infinite number of points of view on motion. After having determined the law that governs this infinite multiplicity one discovers that the Lagrange-Hamilton formulation corresponds to one of them. Four singular and basic points of view are singled out, the others corresponding to more or less complex combinations of the four basic ones. These are developed explicitly and interpreted physically, before deriving the composition laws associated with motion. Thirdly, we examine an important relation of theoretical physics at the frontier of Newtonian, Einsteinian and quantum physics in the light of the present Leibnizian approach. Last but not least, we show how the Leibnizian methodology allows the discovery of an "ontological order" hidden behind the "epistemological disorder" so that a number of controversies may be attenuated or resolved. Then, the relevance of the different Leibnizian distinctions concerning possible worlds, "compossible" ones as well as the actual one (that Leibniz calls the best), are placed in evidence. It becomes obvious that one should distinguish between the "best of all possible worlds" and the "least of possible actions" often related to each other.

I-1. Principle of dynamical relativity.

Conservation properties.

Let us start by specifying what is meant by the properties associated with conservation laws that constitute the heart of dynamics. These conservation laws come from the requirements corresponding to frontal elastic collisions (in one dimension) on which the dynamical framework was, historically, constructed. In this framework, one has the following elementary structure

$E_1 + E_2 = E_1' + E_2'$ $\quad$ $p_1 + p_2 = p_1' + p_2'$ $\quad$ $(E, p : E', p')$ before : after collision

The left hand side is given, while the right hand side is unknown. One needs two such equations to deal with a frontal elastic collision. It is immediately noticed that if E and p correspond to two conservation laws then, it is immediately checked out that $M = a E + b p + c$, where a, b and c are constant coefficients, corresponds also to a conservation law since it verifies $M_1 + M_2 = M_1' + M_2'$. Any other possibility correlating energy to impulse turns out to be unacceptable since it does not verify the above last relation. Such a strong constraint, to which one adds the requirement of getting two and only two conservation laws, will play a major role in the determination of the different dynamical solutions. It should be emphasized here that, contrary to space-time physics inherited from Newton and developed rationally by Lagrange and Hamilton before being extended by Einstein, **Leibniz makes a net distinction between the order of "necessity" and that of "degrees of freedom"; what is imposed by the nature of the problem and what is proposed by our ability to deal with the problem, according to one point of view or another**. On focussing the attention on the necessary requirements (without which dynamics looses its very existence) and adopting the **"principle of plenitude"**, according to which one should consider the maximal number of degrees of freedom compatible with the necessary constraints, one is then, led to an, a priori, infinite number of perspectives on motion (non conserved entities mathematically expressed through internal parameters).

Historical recall of the Principle of dynamical relativity and its Leibnizian extension.

The principle of dynamical relativity goes back (among others) to Huygens, who used it positively to deduce impulse from energy (as shown explicitly in Ref.[1] and implicitly in Refs.[2-4]). A linear combination of the "vis viva" or living force (double of kinetic energy in modern terms) mv^2 with its translated form $m(v + V)^2$ leads to the definition of impulse mv. Indeed, Huygens seems to be the first scientist to assert that if mv^2 corresponds to a conservation law, then the replacement of v by $v' = v + V$ where V is the relative velocity between the two reference frames R and R', leading thus to $mv'^2 = m(v + V)^2$ should also correspond to a conservation law. In order to avoid the redundancy due to the presence of mv^2 into the expression of $m(v + V)^2$, Huygens used a combinatorial procedure by use of the property given in the above paragraph. It asserts that the couple of functions $[mv^2, m(v + V)^2]$ is equivalent to the couple $[mv^2, am(v +V)^2 + bmv^2 + c]$ for any value of the coefficients a, b and c. In spite of the apparent complexity of the second couple, the latter may reduce to mv by an appropriate choice of the different coefficients leading to the couple $[mv^2, mv]$. This simplification (without lack of any generality) is due to the degrees of freedom provided by the different coefficients a, b and c. If one applies any one of the above three forms, one is left with the same final solution. In modern analytical language, this may be accounted for through the notion of the derivative which is a particular linear combination playing here the role of a generator of conservation laws. If one derives mv^2 or its equivalent $1/2mv^2$, in so far as conservation properties are concerned, (since the derivative $\{1/2[m(v + V)^2 - mv^2]\}/V$ is some particular combination to which one adds the consideration of an infinitesimal translation $V\rightarrow 0$), one immediately establishes a **link** between the **procedure proposed by Huygens**, (systematized by Leibniz, the father of infinitesimal calculus) and its **Leibnizian extension, through the derivative notion, to the multiplicity of points of view** which will soon be placed in evidence.

Here lies the key point associated with the "emergent rationality" revived by different modern authors [10, 11, 16]. Let us insist on the remarkable fact that, in its principle, this method is older than the Hamilton-Lagrange formalism. A systematic extension of this idea following a line of thought developed by Leibniz consists in **extending the notion of a derivative so that the latter may account for the possibility of a multiplicity of points of view** on motion. In order to see this more clearly, it should be emphasized that as long as one attaches the notion of velocity to motion, one encounters a serious difficulty since as well-known from the Lagrange-Hamilton formalism, the impulse derives from the Lagrangian ($p = dL/dv$) and not from energy (dE/dv) as proposed here. However, if one adopts the "emergent rationality" recently developed by use of "group theoretical methods" instead of the usual variation formulation based on the "principle of least action", then motion is accounted for dynamically through the notion of rapidity that can be defined and measured without a recourse to any ratio between space and time (as shown by the above mentioned works associated with the "emergent rationality"). Notice that p may derive simultaneously from the Lagrangian as well as from energy, provided that one specifies the parameter with respect to which the derivation is performed (the velocity for the Lagrangian and the rapidity for energy). Both parameters coincide in the degenerate Newtonian parabolic dynamics.

Introduction to the idea of multiplicity of points of view.

The main idea on which Leibniz based his philosophy of nature, asserts that one may look at a given reality (here dynamics) from different perspectives, ordered according to some iterative procedure (recurrent series) allowing one to pass from one perspective to the other. Each would constitute a specific point of view, showing a few remarkable properties, but hiding many others revealed by the other complementary perspectives. To put such a claim into a formal structure, one should replace the unique term v associated with motion by a multiple one, where multiplicity corresponds to the introduction of a Greek index μ as follows: v_μ. Thus, instead of having relations of the usual form: $v = f(E) = g(p)$, one is led necessarily to a multiplicity of relations $v_\mu = f_\mu(E) = g_\mu(p)$, where the conserved entities E and p are accounted for in different complementary manners. However, contrary to what happens in the usual method where no ambiguity occurs in the inversion of g: $p = g^{-1}[f(E)]$, the inversion of g_μ: $p = g_\mu{}^{-1}[f_\mu(E)] = g^\mu[f_\mu(E)]$ leads to a certain ambiguity. One needs to specify that the different points of view correspond to an internal mechanism that has to compensate, in order to yield a unique dynamical relation between the conserved entities p and E. Thus, unlike the other frameworks which do not account explicitly for any multiplicity, one should impose the following invariance property: $g^\mu[f_\mu(E)] = g^\beta[f_\beta(E)]$ whatever the values of the indices μ and β. This fact is essential; otherwise one does not get a predictive theory. The principle of dynamical relativity will be developed following the line of thought developed by Huygens and extended according to the Leibnizian requirements relative to the existence of a multiplicity of points of view on motion (a priori, infinite). This is achieved through the following passage:

$$p = dE/dv \quad \rightarrow \quad p = d_\mu E/dv_\mu = D_\mu\, dE/dv_\mu\,, \qquad D_\mu = D_\mu{}^\mu(v_\mu) \qquad (1)$$

where the linear combination $[f(v + V) - f(v)]/V \Leftrightarrow a\, f(v + V) + b\, f(v)$ with $a = -b = 1/V$ and $E = f(v)$, will be replaced by this other linear but multiple combination:

$$[f^\mu(v_\mu\, T_\mu\, V_\mu) - f^\mu(v_\mu)]/V_\mu \Leftrightarrow a_\mu\, f^\mu(v_\mu\, T_\mu\, V_\mu) + b_\mu\, f^\mu(v_\mu) \text{ with } a_\mu = -b_\mu = 1/V_\mu \text{ and } E = f^\mu(v_\mu).$$

The usual translation v + V is extended as follows: $v_\mu\, T_\mu\, V_\mu$ where the additive composition law is transformed into a number of non additive laws (a priori unlimited) to be determined. The replacement of the well-determined "+" associated with the additive composition law by a multiplicity of undetermined ones noted by T_μ is responsible for the presence of the functions $D_\mu = D_\mu{}^\mu(v_\mu)$ that depend on both the discrete index μ and the continuous parameter v_μ (with possible local discontinuities) associated with the different points of view on motion. [This is developed in detail in Appendix A]. In order to keep in touch with the additive law, one may impose an additional constraint on the new construction so that the additive form appears as one of the different perspectives. Such a constraint imposes that for $\mu = \mu_a$ (a for additive) [with μ_a= a for brevity as explicated later on], one should have $D_\mu{}^\mu(v_\mu) = D_a{}^a(v_a) = 1$ for any v_a. This is obviously verified if one has $D_\mu = D_\mu{}^\mu(v_\mu)$ of the following form : $D_\mu = D_\mu{}^\mu(v_\mu) = [H^\mu(v_\mu)]^{h(\mu)-h(a)}$ where it is readily checked that for $\mu = a$ one gets automatically $D_a = D_a{}^a(v_a) = 1$ for any v_a . Such a scale law deduced in Appendix B will play a major role in the ordering of the infinite multiplicity of points of view on motion.

The justification of the operator $d_\mu/dv_\mu = D_\mu\ d/dv_\mu$, its specification by use of symmetry considerations associated with energy and the major roles it plays are developed respectively in the Appendices A, B and C that one does not need to consult in a first reading. The important thing to know is that this operator allows dealing with the idea of translational motion at the basis of the relativity principle in a general framework (qualitative and quantitative) including an infinite number of points of view. It is also worth noting that it allows dealing with a discontinuous framework where this operator constitutes an "absorber" of discontinuities as shown in Appendix C. In order to avoid any ambiguity, let us note that the desire to go beyond the continuity hypothesis is not arbitrary but it is motivated by a physical necessity associated with the first rejected quantitative dynamical model that one encounters in 17^{th} century dynamics. The latter does not enter into the mould of analytical methods so that its rejection by use of these methods is logically untenable. As to its rejection on an empirical ground in reason of its incompatibility with experimental measurements, this may be accepted only locally as emphasized by Leibniz since an approach may be valid at one scale and invalid at another, the first well-known example being that of Newtonian dynamics.

Subjective qualitative and trans-subjective quantitative versions of dynamical relativity.
Having placed in evidence the double faceted role played by the new operator $d_\mu/dv_\mu = D_\mu\ d/dv_\mu$ and accounting for the different potential subjective measures associated with motion, we shall express the "dynamical relativity principle" in two different versions:(i) a subjective **qualitative version** constituted of an infinite number of points of view on motion and (ii) a trans-subjective **quantitative version** whose formal structure lies beyond any subjective point of view. The quantitative character of the trans-subjective version is a direct consequence of a mechanism of compensation, which eliminates all the subjective undetermined (qualitative) points of view. This is a typical feature of the Leibnizian methodology that allows **obtaining a quantitative relation** between the conserved entities (energy and impulse) **without postulating any quantitative property relative to motion**. This character is essential to the present Leibnizian dynamical unifying framework apt to include potentially an unlimited number of points of view on motion. Such a distinction between qualitative and quantitative versions has no counterpart in conventional formulations, since all of them start by defining motion quantitatively in one way or another. These two qualitative and quantitative versions are given explicitly in the forthcoming developments.

(i) Subjective qualitative version of the "dynamical relativity principle".

The main idea lying behind the present expression of the "dynamical relativity principle" concerns the necessity of having two conserved quantities where the second is deduced from the first through the above-mentioned extended derivation procedure (that we also call a μ-derivation in order to specify the multiplicity attached to this kind of derivation). Since the derivative plays here the role of a generator of conservation laws, one should impose a constraint on the second order extended derivative preventing thus the unlimited multiplicity of conserved quantities. This way of thinking is typically Leibnizian where no recourse to experiment is needed. Unlike Huygens method starting with mv^2 (or better with mw^2 distinguishing thus, the velocity v from the rapidity w), before deriving impulse $p = dE/dw$, Leibniz's formulation defines inertia as follows: $I = d^2E/dw^2 = dp/dw = m = c^{te}$ (with $w = 0$, $p = 0$: state of rest) where the second derivative associated with inertia I corresponds to a constant having the dimension of a mass. This procedure has the merit to deduce both conserved entities E and $p = dE/dw$. **The definition of inertia I corresponds here to the constraint imposed in such a way that one leads to only two conserved entities**. It is easily shown that any other derivation does not depend on w and cannot be associated with a conservation law dealing with the elastic collision problem at the basis of the construction of dynamics. This procedure, forgotten for centuries, constitutes the basis of what we have called the "emergent rationality". The latter has been developed recently in the framework of Einstein's dynamics, where inertia I (the constraint) turns out to be associated with energy (a variable entity) rather than with mass (a constant entity) obtaining thus $I = d^2E/dw^2 = E/c^2$. If the successive derivations do not vanish as in the first case, these do not provide other conserved entities since one gets alternatively E and p (up to a multiplicative factor). More generally, if one associates inertia neither with mass nor with energy but with the general form compatible with conservation properties which combines energy and impulse as follows: $I = \lambda E + \gamma p + \eta$, (as shown in the above paragraph associated with conservation properties, one is left with: $I = d^2E/dw^2 = \lambda E + \gamma p + \eta = \lambda E + \gamma\, dE/dw + \eta$ with $p = dE/dw$. Recalling that one of the cornerstones of the Leibnizian methodology consists in the inclusion of a multiplicity of points of view on motion, one has to replace the simple derivative by the extended one leading to the following final form expressed through a second order differential system of equations as follows:

$$I = d_\mu{}^2E/dv_\mu{}^2 = \{D_\mu\, d/dv_\mu\, [D_\mu\, d/dv_\mu]E\} = \lambda E + \gamma p + \eta \qquad (2)$$

Where p is defined through Eq.(1). **Here is the most general expression of the "dynamical relativity principle"** accounting simultaneously for an a priori infinite number of points of view on motion. Eq.(2) extends Huygens method rediscovered and associated with the "emergent rationality" proposed by different authors. *In particular, C. Comte wrote several papers on the subject matter one of which is directly linked to Leibniz's methodology. In these papers as well as in other ones recalled in Appendix J, one encounters the following expression:* $d^2\varepsilon(x)/dx^2 = \sigma^2\varepsilon(x)$ *or equivalently* $d^2E(w)/dw^2 = 1/c^2\, E(w)$,*(*$cx = w$*: rapidity and* $\varepsilon\, mc^2 = E$*: energy) which is a particular form of Eq.(2). It corresponds to the two following restrictions:* $\gamma=0$, $\eta=0$ *and* $\mu = a$ *with* $D_a = 1$ *so that neither the most general case associated with the properties of conservation laws nor the infinite multiplicity of points of view on motion are considered. One should emphasize that in the present approach we wish to be as close as possible to Leibniz's conceptual and mathematical (differential calculus) considerations.*

As long as one uses a specific perspective as in the usual formulations, and does not consider a framework capable of including simultaneously different points of view on which one may perform formal operations, one is condemned to deal with simple models and not with a real (Leibnizian) theory. Only such a theory is capable of deriving the different solutions produced by the presently available dynamical models in a unified manner. Another main feature that distinguishes the present Leibnizian dynamical framework from the ones derived in the usual models (as in Refs.[10-21]) is relative to possible extensions in different directions. These are developed later on among which a finite class of dynamics is underlined for which energy remains bounded. Another generalized dynamics (analyzed in the second part of this work) is associated with a framework where parity (usually attached to isotropy) is broken ($\gamma \neq 0$). This is not obtained in an ad hoc manner but in a systemic way through the “trans-subjective” version of the dynamical relativity principle, developed hereafter.

(ii)Trans-subjective quantitative version of the « dynamical relativity principle ».

In order to pave the way for new extensions (developed in the second part of this work) we shall develop the “trans-subjective quantitative procedure” associated with the “dynamical relativity principle”. To this end, let us recall the following operational properties (see end of Appendix A).

$$d_\mu{}^2R/dv_\mu{}^2 = \{D_\mu\, d/dv_\mu\, [D_\mu\, d/dv_\mu]R\}= p^2\, d^2R/dE^2 + p\,(dp/dE)\, dR/dE \qquad (3,\ a)$$

whose application to energy (R = E) leads to a passage from a second order structure to a first order one as follows

$$d_\mu{}^2E/dv_\mu{}^2 =\{D_\mu\, d/dv_\mu\, [D_\mu\, d/dv_\mu]E\} = p\,(dp/dE) \qquad (3,\ b)$$

Here lies the key point allowing to pass from a qualitative result associating E with v_μ to a quantitative one relating E to p. The qualitative character is due to the presence of the undetermined functions $D_\mu = D_\mu{}^\mu\,(v_\mu)$ corresponding to v_μ that disappear in favour of impulse. It should be emphasized that the couple (E, p) is basic and objective contrary to the different points of view that may be chosen in one way or another according to the measurement one uses to account for motion. If motion can be linked to energy and impulse in different manners as follows: $v_\mu = f_\mu(E) = g_\mu(p)$ one should keep in mind that the relation between energy and impulse is unique (otherwise one would not have a predictive formulation). On assuming that the functions g_μ can be inverted ($g_\mu\, g^\mu$ = Id) then one may write $p = g^\mu[f_\mu(E)] = G(E)$ where the Greek indices should disappear through a compensation procedure. This mechanism of compensation is accounted for through the trans-subjective version of the dynamical relativity principle where the different points of view on motion v_μ are eliminated in favour of the impulse p. (One may refer to Appendix G for more details).

At this point, it is worth noting that the passage from v_μ (motion: non conserved entities) to p (impulse: conserved entity) has its counterpart in the Lagrange-Hamilton formalism through the so-called Legendre transformation. However, in this case one deals uniquely with a quantitative framework, there is no possible passage from a qualitative one when dealing with motion to a quantitative one when dealing with impulse as shown through the combination of (3, b) with (2) leading to $pdp/dE = \lambda E + \gamma\, p + \eta$.
This equation is obviously quantitative while its subjective counterpart:
$d_\mu{}^2E/dv_\mu{}^2 =\{D_\mu\, d/dv_\mu\, [D_\mu\, d/dv_\mu]E\}== \lambda E + \gamma D_\mu dE/dv_\mu + \eta$
is not, since the functions D_μ are still indeterminate.

<u>A second equivalent but more explicit procedure.</u>

Instead of using the above mentioned transformations given in (3, a) and (3, b), one may also proceed by decomposing Eq.(2), using (1), and obtaining thus the two following first order differential equations

$$D_\mu \, dE/dv_\mu = p \qquad D_\mu \, dp/dv_\mu = \lambda E + \gamma p + \eta \tag{3, c}$$

On considering their ratio, one eliminates all subjective elements leading to what we call the "trans-subjective" version of the dynamical relativity principle which takes on the following form

$$I = pdp/dE = p\,p' = \lambda E + \gamma p + \eta \tag{4, a}$$

This equation reduces to

$$I\,(E) = pdp/dE = p\,p' = \lambda E + \eta \tag{4, b}$$

if one imposes the usual isotropy requirement where Eq.(4,a) should remain invariant when $p \to -p$ so that one gets $\gamma = 0$. Notice that the passage to three dimensions is immediate where pp' is to be replaced by the following scalar product **p.p'** with **p** = (p_1, p_2, p_3). Notice finally that the two constants λ and η may be eliminated by a simple derivation procedure leading to

$$p\,p''' + 3\,p'\,p'' = 0, \quad p' \equiv dp/dE \;\Leftrightarrow\; p^2 = U(E) \quad \text{such that} \quad U'''(E) \equiv d^3U/dE^3 = 0 \tag{4, c}$$

U(E) is a function of E whose third derivative should vanish otherwise the principle of dynamical relativity is violated in the particular case of $\gamma = 0$. Notice that the above-mentioned invariance associated with the first order differential equation I(E) is preserved in the fundamental relation U(E) between energy E and impulse p since the replacement of p by – p keeps the function U(E) invariant. This quantitative constraint associated with the vanishing of the third order derivative will play a major role in the study of new dynamical frameworks developed in so-called "doubly-special relativity" as shown later on. As to the more general equation given in (4, a) or its subjective version given in (2), it will be dealt with in order to examine the solutions associated with an uneven framework where the symmetry associated with energy (parity requirement) is broken. This equation may also be written in a form such that the coefficients λ, γ, and η disappear from the differential equation and may be recovered through the constants of integration. This is done in two steps: first we deduce from (4, a)

$$p\,p''' + 3\,p'\,p'' = \gamma\,p'' \quad \Leftrightarrow \quad p\,p'''/p'' + 3\,p' = \gamma \tag{4, d}$$

by analogy to (4, c), then we eliminate the coefficient γ by a simple derivation as follows:

$$\mathbf{p'p''p''' + p\,p''p'''' - p\,p'''^2 + 3\,p''^3 = 0, \qquad p' = dp/dE} \tag{4, e}$$

This last fourth order differential equation is the most general equation associated with trans-subjective version of the dynamical relativity principle. It lies beyond the consideration of the finite and the infinite, the closed and the open, the straight and the curve, the continuous and the discontinuous, the analytic and the non-analytic… Such a general framework including different solutions belonging to one of the above considerations is a direct consequence of the degrees of freedom provided by the possible variety of the limit conditions needed to determine the integration constants associated with the above fourth order differential equation. These integration constants will permit to define a "broken parity" constant γ, an invariant mass m, a first coupling constant c (velocity of light or upper limit velocity) and a second one E_M that may be associated with an upper limit energy (as shown in the second part of this work).

Comments on simplicity, generalization by differentiation and account for discontinuous solutions and partial dynamical frameworks.

We start by concentrating on some aspects whose details are given in the second part of Appendix K where the above most general equation (4, e) is discussed and compared to (4, c). In particular, it is shown that contrary to the integration of (4, c) that leads to the following relatively simple expression: p^3 p'' = A, the integration of (4, e) or its equivalent (4, d) leads to a more complicated form expressed by $p^{f(\gamma,p')}$ p'' = A exp[$\gamma \int$ g(p, p', p")dE] with f(0, p') = 3. It is easily seen that one recovers p^3 p'' = A when γ = 0. Such a possibility does not occur in the "usual" or "emergent" rationalities since in both cases, the even character of energy is considered to be primary. Here, not only this criterion is not necessarily imposed but when the latter is postulated, this is done after the relativity requirement. This allows a better understanding of the different steps that lead to the final solution. This mechanism of **generalization by differentiation** which is typically Leibnizian, leads to another key point attached to the **continuity hypothesis**. It allows dealing with the "dynamical relativity principle" in a framework which is **not necessarily analytical and continuous**. Such a property is useful and natural from both mathematical and physical standpoints. In so far as mathematics is concerned, one should recall that a differential equation may have analytical or non analytical solutions according to the imposed limit conditions. The first historical partial dynamical formulation due to Descartes and founded on conservation properties turns out to be non analytical. Thus, the only way to judge its compatibility with the dynamical relativity principle is to embed it into an appropriate framework. Otherwise, any judgement lacks rationality. Notice finally that because of its independence of motion, a differential equation such as (4, c) or (4, e) which constitute quantitative compact expressions of the "dynamical relativity principle" may be used to verify the compatibility of some incomplete dynamical frameworks. [In the first dynamical approaches developed in the 17^{th} century where the idea of positive substance was predominant in comparison to that of matter located and mobile in space and time, partial system of equations were proposed in which motion was not sufficiently well-defined but only the relation between the conserved entities was specified. This is precisely the case of the Cartesian statement according to which the modulus or absolute value of impulse (positive active substance) is a conserved entity]. The "trans-subjective procedure" deals precisely with the dynamical relativity principle in the absence of any quantitative consideration on motion. This allows judging the relevance of such partial dynamical frameworks.

Trans-subjectivity and Legendre transformation.

This paragraph is devoted to the development of some remarkable properties among which those associated with trans-subjectivity and its relation to Lagrange-Hamilton formalism through the Legendre transformation, responsible for the expression of energy in terms of impulse instead of velocity. To this end, it may be useful to show that Eq.(4, d) may also be expressed as follows

$$\gamma = p\ \{Z^{-1}\ (dZ/dE)\} = p\ dQ/dE\ ,\quad Z = p^3\ p''\quad Q = \mathrm{Ln}[A\ Z],\qquad p'' = d^2p/dE^2 \qquad (4,\ f)$$

Ln[x] indicates the natural logarithm of x whose derivative is defined up to a multiplicative constant ({Ln[ax]}' = {Ln[x]}'). In order to pave the way for a possible comparison with the Lagrange-Hamilton formalism and more particularly with the first canonical Hamilton equation for a free particle (where the velocity concept is expressed through the derivative of energy with respect to impulse û = dE/dp) one should express the above equation in such a way that it is written in terms of derivatives with respect to impulse p, instead of energy E. This inversion procedure leads to:

$$\gamma = (p/\hat{u})\,(S^{-1}[dS/dp]) = (p/\hat{u})\, dR/dp\ ,\ S = (p/\hat{u})^3\, d\hat{u}/dp \quad R = \mathrm{Ln}\,[B\,S] \qquad (4, g)$$

after having used the following inversion properties and notations:

$$p' = (dp/dE) = 1/(dE/dp) = 1/\hat{u}\ , \quad p'' = d^2p/dE^2 = -\,(d^2E/dp^2)/(dE/dp)^3 = -\,(d\hat{u}/dp)/\hat{u}^3 \qquad (4, h)$$

At this point, let us note that in the same manner as the subjective version of the principle of dynamical relativity singles out one natural point of view known as rapidity (at the basis of the emergent rationality, that corresponds to $D_\mu = D_a = 1$) as discussed earlier, the trans-subjective version which corresponds to what we are developing in this Section singles out the parameter û which will play a major role. In spite of the difference between the two parameters w and û associated with dE = pdw and dE = û dp these possess the same physical dimension. In particular, û turns out to be associated with the first Hamilton canonical equation for free particles (velocity). The latter is obtained after the application of a Legendre transformation on the Lagrangian formalism, obtaining thus energy (also called the Hamiltonian) in terms of impulse instead of velocity. In spite of the fact that the present trans-subjective procedure eliminates the different points of view on motion in favour of impulse, the resulting differential statement of this version of the principle of relativity singles out p' whose inverse corresponds to the dynamical definition of the velocity, at least in the frameworks of Newtonian and Einsteinian dynamics where one has: dE/dp = dx/dt. (Such an equality is not valid anymore in some recent dynamical approaches as shown in Refs.[8,16]). Thus, the notion of velocity associated with the first canonical Hamilton equation plays a singular role as compared to the different other points of view on motion. This interpretation will be developed more explicitly later on, in the present work. [We simply note here that if one sets u = p/m, û = dE/dp and v = p/M (M = m and M = E/c^2 in Newton and Einstein dynamics respectively), then Newton's dynamics leads to u = û = v, Einstein's dynamics to u ≠ û = v and the present Leibnizian dynamics to u ≠ û ≠ v].

<u>*Integration and change of variables.*</u>

The expressions (4, f) and (4, g) are easily integrated for $\gamma = 0$, so that one is left with

$$p^3\, p'' = p^3\, d^2p/dE^2 = -\,C \Leftrightarrow (p/\hat{u})^3\, d\hat{u}/dp = C \qquad (4, i)$$

A second integration leads to

$$1/\hat{u}^2 - C/p^2 = B \qquad ,\ \hat{u} = dE/dp \qquad (4, j)$$

where B and C are the two integration constants. It will be shown later on that, according to the choice of the couple of integration constants: (C, B), one gets different sorts of dynamics. It should be emphasized that the interpretation of the constants C and B may differ from a dynamical framework to the other, according to whether one is dealing with infinite or finite approaches. The infinite ones are those for which energy may tend to infinity (as for Newtonian and Einsteinian dynamics) and finite approaches are those for which energy remains finite (as for some recent dynamical frameworks dealing with so-called "doubly or deformed special relativity" [7-8]). This is developed explicitly in the second part of this work.

[It is worth noting that the isotropic ($\gamma = 0$) dynamical relativity principle where parity is satisfied, greatly simplifies if one expresses it through the following change of variables:

$$x = 1/p^2 \quad y = 1/\hat{u}^2 \tag{4, k}$$

since in this case Eq.(4, j) reduces to

$$y = Cx + B \Leftrightarrow dy/dx = C \Leftrightarrow d^2y/dx^2 = 0 \tag{4, l}$$

When γ does not vanish (anisotropy or more generally broken parity), one shows that the substitution of (4, k) into (4, g) leads to

$$\gamma = -2\,\varepsilon\, x\, y^{1/2}\,(d^2y/dx^2)/(dy/dx) \Leftrightarrow d^2y/dx^2 + [\gamma/(2\,\varepsilon\, x\, y^{1/2})](dy/dx) = 0, \quad \varepsilon = \pm 1 \tag{4, m}$$

These different expressions of the « trans-subjective » version of the dynamical relativity principle will play a major role at different levels in the present work].

Leibnizian necessity and degrees of freedom.
The lack of determination of what is meant by motion is illustrated by the Leibniz's multiplicity of points of view, where motion does not belong to the realm of necessity as for conservation laws. Contrary to such laws whose absence implies the absence of dynamics itself, motion can be introduced in one way or another according to the point of view one adopts on it. As it will be shown in the next section, if one admits that the study of an elastic frontal collision implies necessarily the existence of two and only two conserved entities, the number of ways through which one may account for motion to deal with these two entities remains open to infinity, at least in principle. Those who adopt exclusively the "usual rationality" of Lagrange-Hamilton formalism do not recognize the Leibnizian distinction between what amounts to **necessity** and what amounts to **degrees of freedom**. This is due to the fact that the Lagrange-Hamilton formalism is founded on a too restrictive framework that imposes a non necessary requirement (the velocity concept) from the start. This reasoning also applies to the "emergent rationality". All who adhere to an "exclusive approach" that do not distinguish between what **can be done** (introduction of velocity, rapidity or still other points of view) and what **should be done**, (necessity of two conservation laws), think that Leibniz's "inclusive approach" constitutes a contradictory context and leads to an irrational thought inappropriate to physical investigations. This is due to the absence of a general rational inclusive framework, developed in this work that supports Leibniz propositions. The construction of such a framework on a rational ground will

show that Leibniz was right when his conciliatory attitude led him to consider that Descartes, Huygens and Newton dynamics may be locally valid. There is no contradiction in adopting different co-existent perspectives on dynamics provided that they respect the realm of necessity or essence (Aristotle's entelechy: here the conservation laws) without which the construction of a predictive dynamics becomes an impossible task. The incapacity of dealing properly with prenewtonian dynamics through the "usual and emergent rationalities" will be shown in the third part of this work.

I-2. Extension of Einstein's dynamics including different perspectives on motion among which the Hamilton-Lagrange point of view.

In the forthcoming developments we shall show how to deal with an infinite multiplicity of points of view on motion. The procedure will be applied to Einstein's dynamics, whose historical foundation uses experimental evidence at least indirectly through electromagnetism, or can be presented through the extension of Newton's dynamics, where simultaneity is revisited, so that the concept of absolute time does not hold anymore. The present approach allows one to express Einstein's equivalence relation $E = Mc^2$ [a particular case of $M \equiv I = \lambda E + \gamma\, p + \eta$ given in (2)] in the following form justified in the Appendix B through (B1) and (B14).

$$M = d_\mu{}^2E/dv_\mu{}^2 = \{d_\mu/dv_\mu\, [d_\mu/dv_\mu]E\} = \{D_\mu\, d/dv_\mu\, [D_\mu\, d/dv_\mu]E\} = E/c^2 \qquad (5, a)$$

with

$$p = d_\mu E/dv_\mu = D_\mu\, dE/dv_\mu \ , \qquad (5, b)$$

such that

$$D_\mu = D_\mu{}^\mu(v_\mu) = D^\mu(v_\mu\, , \mu) = \{D[f^{\,\mu}(v_\mu)]\}^{\,a-\mu} = [D(E/E_0)]^{a-\mu} \qquad (5, c)$$

We recall that the Greek exponent **μ** may take any value while $\mu_a = a$ (for convenient notations) corresponds to a fixed value that one will choose numerically later on so. (The different points of view v_μ correspond to v_1, v_2 , v_3 etc. The numbering 1, 2, 3…is a question of convenience. If one does not make appropriate choices of the degrees of freedom provided by the underlying mathematical structure, then one may lead to cumbersome notations where the above simple order is lost so that one gets not only negative numbers but also non-integer ones).

In order to get well determined solutions, one adds the following local conditions

$$p = 0,\ \ v_\mu = 0\ , \quad E = E_0 \qquad \forall\, \mu \qquad (6)$$

One may replace here the local conditions by global ones associated with the **even character for energy and odd one for impulse**. Here, the local constraints given in (6) turn out to be equivalent to the global ones expressed through symmetry requirements (this is not true in general). This is due to the fact that Einstein's dynamics, does not correspond to the general case $M \equiv I = \lambda E + \gamma\, p + \eta$ but only to $M = \lambda E$ where the coefficient responsible for parity breaking or anisotropy γ is absent from Eq.(5, a).

The inertia $I \equiv M$ which has the dimension of mass (not to be confused with invariant mass not yet defined) is obtained through the second extended derivative including an infinite number of equations each one corresponding to one specific point of view on motion. It plays the role of a constraint imposed on dynamics to get two and only two conservation laws. Indeed, let us recall that in the absence of such a constraint and since each μ-derivation leads to one conservation law one gets an infinite number of such laws leading to an ill-posed problem. If the infinite number of "subjective" degrees of freedom is, a priori, possible (according to Leibniz's "principle of plenitude": consideration of all non contradictory possibilities) any infinity associated with "objective" elements (conserved entities) leads to the absence of a well-determined predictive approach.

Determination of the law governing the infinite multiplicity of points of view.
It is worth noting that the system of equations given in (5) is not yet complete because of the undetermined form associated with Eq.(5, c). The concepts of energy and impulse have already been defined: this is not the case for the mass concept since it is independent of the relativity principle, which operates on variable entities and not on constant ones. To this end, one notes that among the, number of points of view on motion, each of which corresponding to an odd function when expressed in terms of impulse, one may single out one of them [$\mu = \mu_p$ that we identify to p for the simplicity of notations: $\mu_p = p$] in such a way that it verifies a simple proportionality relation (which obviously satisfies the odd symmetry) and allows one to associate the mass concept with the proportionality coefficient. Thus, one sets

$$p = m\, v_p \quad \text{(p for proportional)} \tag{7,a}$$

It is worth noting that such a constraint is sufficient to get a well posed physical problem and a predictive theory. Indeed, on combining (7, a) with (5, a) and choosing the point of view of order "p" such that $\mu_a - \mu_p = a - p = 1$ for convenience and with no lack of generality, then, one gets

$$M = [D(E/E_0)]^{a-p} dp/dv_p = D(E/E_0)\, m = E/c^2 \;\Rightarrow\; D(E/E_0) = E/mc^2 \tag{7,b}$$

Since c is a constant introduced in (5, a) for dimensional homogeneity and can be chosen arbitrarily, then one immediately notices that the following choice

$$E_0 = mc^2 \qquad \Rightarrow \qquad D(E/E_0) = E/E_0 \tag{7,c}$$

command attention. The substitution of (7, c) into (5, c) leads to the following final result

$$D_\mu = D^\mu(v_\mu , \mu) = [f^\mu(v_\mu)]^{a-\mu} = (E/E_0)^{a-\mu} = (E/mc^2)^{a-\mu} \tag{8}$$

It is immediately checked out that the substitution of (8) into (5, a) leads to a well-determined system of second order equations composed of an infinite number of relations each one constituting a point of view on motion exactly as required by Leibniz.

Before the end of this paragraph, let us note that Eq.(7, a) may be associated with the Newtonian "fundamental equation of dynamics": $F = m\,\Gamma$. If one interprets F as the time derivative of

impulse p and Γ as the time derivative of v_p then one gets a direct link with the Newtonian concepts. It should however be noted that the latter equation appears only at the end of the dynamical structure in view of completing it. Notice that this constraint is not needed when dealing with the case corresponding to $\mu = a$. In this unique singular situation, with or without the knowledge of $D(E/E_0)$, one gets $D_\mu = [D(E/E_0)]^{a-\mu} = 1$ for $\mu = a$.
Because of the importance of the "usual rationality" based on Lagrange-Hamilton formalism, let us see how is it articulated in the present multiple approach and how does it appear to be one of the points of view.

Deduction of the Lagrange-Hamilton structure associated with one specific point of view.
The attention will be drawn firstly on the natural emergence of the Lagrangian in the study of one of the points of view, in relation to the "subjective" version of the dynamical relativity principle. Secondly, it is shown that the velocity concept (in addition to its subjective character) appears naturally in the "trans-subjective" version of the dynamical relativity principle.

a) "Subjective" version of the principle of dynamical relativity (economy of thought versus structural simplicity).
The resolution of the system of equations given in (5, a) runs as follows. In a first step, the substitution of (8) into (5, a) leads to a well determined system of equations where energy E may be obtained in terms of the points of view on motion denoted by v_μ. In a second step, one deduces the impulse from Eqs.(5, b) subject to (8). Among the infinite number of points of view, one discovers that the one associated with

$$\mu = a + 2 \qquad (9)$$

satisfies the following properties (see Appendix F for more details).

$$p = dL/dv \qquad E = v\, dL/dv - L \qquad (10)$$

which constitute the basic equations of Lagrange-Hamilton formalism for a free particle, where the Lagrangian L is given by

$$L = -E_0^2/E \qquad v_{a+2} = v \qquad (11)$$

The expression of L appears in a natural way through the resolution of the second order differential equation associated with (5) subject to (8) and (9). More precisely, the solution of this equation requires a change of variable for its integration. This change of variable corresponds to the one given in (11) and that coincides with the so-called Lagrangian of a dynamical system since it verifies (10). To see this, let us note that the resolution of the following equation:

$$F(E^*E^{*''}, E^{*'2}, E^{*\,6}) = E^*E^{*''} - 2\,E^{*'2} - E^{*\,6} = 0$$

deduced from (5, a), (8) and (9) after having used non dimensional notations:

$E^* = E/mc^2$, $v^* = v/c$ and $E^{*'} = dE^*/dv^*$

requires the following change of variable ($L^*E^* = -1$). Thus, we are led to a simpler form

$$G(L^{*''}, L^{*-3}) = L^{*''} + L^{*-3} = 0, \qquad L^{*'} = dL^*/dv^*,$$

whose integration is elementary and easy to obtain. It corresponds to $L^{*2} + v^{*2} = 1$ where L^* corresponds to the Lagrange function of Einstein's dynamics (when expressed in a non-dimensional form).

The present Leibnizian approach provides precious information concerning the significance of the Lagrangian which has the dimension of energy, without corresponding to a conserved entity. Its "raison d'être" lies in the simplicity with which one deduces impulse ($p = dL/dv$). Thus, the lack of "**economy of thought**" when dealing with the Lagrangian (unnecessary to dynamics as shown in the present Leibnizian approach) is somehow compensated by a certain "**structural simplicity**". The derivation of p from $p = dL/dv$ is obviously simpler than $p = d_\mu E/dv_\mu = D_\mu dE/dv_\mu$.

b) "Trans-subjective" version of the principle of dynamical relativity (relation to the first canonical Hamilton equation).

The use of a Legendre transformation allowing to express energy in terms of the impulse p instead of the velocity concept v, consists in differentiating (10) so that one gets

$$dE = d(vp - L) = vdp + (p - dL/dv)\, dv = vdp \Rightarrow v = dE/dp$$

corresponding to the so-called "first canonical Hamilton equation".

The velocity concept which corresponds to the point of view of order $\mu = a+2$ as shown in (9)-(11) constitutes a singular point of view. Its singularity lies in the fact that it is intimately related to the "trans-subjective" version of the principle of dynamical relativity whose main object is to express this principle without recourse to any particular and specific point of view. However, since this principle is, in the isotropic case, basically expressed through the following differential form

$$pp''' + 3\,p'\,p'' = 0$$

it is then directly related to v through p' or more directly through $\hat{u} = dE/dp$ since $\hat{u}$ coincides with v (as shown from the comparison of the above result obtained through the Legendre transformation and the one associated with the "trans-subjective" version of dynamical relativity $(4, h)_4$). If one uses the compact form deduced from (4, j) and (4, k), one may write

$$d^2y/dx^2 = 0 \Rightarrow dy/dx = C \Rightarrow y = Cx + D \Leftrightarrow 1/\hat{u}^2 = 1/v^2 = C/p^2 + D = m_d^2/p^2 + 1/v_d^2$$

$C = m_d^2$ and $D = 1/v_d^2$ are introduced for dimensional homogeneity. At this point, one discovers that the following well-known relativistic form linking dynamics to kinematics (space-time metric)

$1/v^2 - 1/u^2 = 1/c^2 \Leftrightarrow dt^2/dr^2 - d\tau^2/dr^2 = 1/c^2 \Leftrightarrow c^2dt^2 - dr^2 = c^2d\tau^2$

and associated with the following definitions:

$v = dr/dt \quad u = dr/d\tau \quad p = m\,u$

constitutes a particular form of the dynamical relativity principle.
One should recall that the above expressions are directly deduced from the particular isotropic case. Thus, the constants $C = m_d^2$ and $D = 1/v_d^2$ are to be identified with m^2 (m: invariant mass) and with $1/c^2$ (c: light velocity or upper limit velocity) only when dealing with Einstein's approach. This is not necessarily the case if one looks for a dynamical finite framework where energy and impulse do not tend to infinity as shown later on in the extended framework comparable to the one encountered in so-called "doubly special relativity" also named by "deformed special relativity".

If the approach proposed here remains compatible with the Lagrange-Hamilton formalism, it nevertheless corresponds to a wider framework (in so far as dynamics is concerned) since it includes it as one point of view among others. However, this formalism, included in the Leibnizian structure, possesses a number of remarkable properties (as shown in Appendix F) and constitutes one singular point of view among the four basic ones. These will be derived from the second order differential system given in (5) subject to (8) developed in the next paragraph.

Emergence of four singular and basic points of view on motion.
In order to keep focussing the attention on the physical points and not on the mathematical aspects, we simply explain how to proceed. Instead of dealing with the second order system of equations directly as proposed in the previous paragraph, one may combine (5,a) with (5,b) obtaining thus the two following first order differential equations (trans-subjective procedure) that lead to

$$p\,dp/dE = E/c^2 \quad \Rightarrow E^2 - c^2p^2 = E_0^2 \qquad (12)$$

and

$$c\, dv_\mu /dp = (E/E_0)^{(1-\mu)} \qquad (13)$$

from which one deduces the following non dimensional expression:

$$x_\mu = \int [1 + X^2]^{(1-\mu)/2} dX \qquad X = p/mc = cp/E_0 \qquad x_\mu = v_\mu /c \qquad (14)$$

We have preferred here the notation x_μ to v^*_μ for simplicity. Only four points of view turn out to be singular and basic, the others being constituted by more or less complicated combinations of these four basic points of view. In these calculations the constant "a" which appears in Eq.(6) has been identified to 2 for convenience since in this case the four points of view turn out to be ordered from 1 to 4. Thus one may deduce from (14) and (12) the following equations: [one may refer to Appendix H for more detail: (H13)-(H16)].

$$X = cp / E_0 = x_1 = \sinh x_2 = \tan x_3 = x_4 / [1–x_4^2]^{1/2} \quad (15)$$

$$Y = E/ E_0 = [1+x_1^2]^{1/2} = \cosh x_2 = \sec x_3 = 1 / [1–x_4^2]^{1/2} \quad (16)$$

where x_μ , $\mu = 1$ to 4, constitute the above mentioned four basic points of view each of which is associated with one specific measure. The first, second and fourth points of view correspond to celerity, rapidity and velocity if one adopts the three denominations given by the synthetic paper of Ref.[13]. As to the third point of view that one may call mobility, it corresponds to an interpretation of length contraction in terms of a rotation.

Composition laws associated with motion.

Among the different composition laws only the one associated with the rapidity x_2 constitutes an additive parameter, the others can be easily obtained through a simple transformation by use of the following relations

$$x_\mu = \int [\cosh x_2]^{2–\mu}\, dx_2 \quad \Rightarrow \quad x_\mu° = \int [\cosh x_2°]^{2–\mu}\, dx_2° \quad (17,a)$$

where x_μ are measured in a reference frame R while $x_\mu°$ are measured in a translated reference frame R° such that

$$x_2° = x_2 + X_2 \quad \Rightarrow \quad x_\mu° = x_\mu\, T_\mu\, X_\mu \quad (17,b)$$

The relation given in (17,a) is deduced from (14),(15) and (16) where one expresses X as a function of x_2 because of its additive character. In particular, one may single out the following three relations

$$x_1 = \sinh x_2 \qquad x_3 = \mathrm{Arctan}[\sinh x_2] \qquad x_4 = \tanh x_2 \quad (17,c)$$

In order to perform a link with the usual dynamical approach one easily deduces from (17,c) and (17,b) the velocity composition law

$$x_4° = x_4\, T_4\, X_4 = [x_4 + X_4] / [1 + x_4 X_4] \quad (17,d)$$

More details are given in Appendix M through (M1)-(M5) concerning the question of the different composition laws associated with motion and particularly those attached to the usual space-time physics.

Canonical Hamilton equation for a free particle examined through the present Leibnizian methodology.

If one looks for only one relation that may articulate Newtonian, Einsteinian and quantum physics in a unified way, one discovers that such a formula exists, and corresponds to the so-called first canonical Hamilton equation for a free particle. Indeed, this equation expressed by $dx/dt = dE/dp$ may be cast in a weaker form so that one deduces the following relations

$v \equiv \Delta E/\Delta p = \Delta x/\Delta t \qquad\qquad A \equiv \Delta E\ \Delta t = \Delta p\ \Delta x$

where the differential forms have been replaced by finite differences. The first of these equations associated with motion, distinguishes between Newtonian and Einsteinian dynamics through the concept of velocity v which is bounded (Einstein) or unbounded (Newton). The second of these associated with the concept of action A whose **inverse** is also bounded or unbounded according to whether one deals with classical (unbounded) or quantum physics (bounded through Planck's constant).

The above mentioned first canonical Hamilton equation plays also a major role in the mechanism associated with particle-wave duality. The de Broglie proportionality relations between energy and frequency on the one hand then between impulse and wave number on the other hand associate a wave nature with concepts usually considered to deal with particles. Thus, one gets $dE/dp = d\omega/dk = V_g$ and $E/p = \omega/k = V_\phi$ from which one deduces the following well-known relation $V_g V_\phi = c^2$ associated with Einstein's dynamics when interpreted in the framework of a wave like picture dealing with group and phase velocities, noted by V_g and V_ϕ respectively. These considerations intimately related to motion through the first canonical Hamilton equation, play such a central role in physics that they deserve to be examined and connected with the present Leibnizian approach. Let us recall that one of the main goals of this approach is to extend motion so that it may be accounted for through different points of view, where the velocity concept, directly articulated to the Hamiltonian formalism as shown above, is but one point of view among others. In order to see this, one should recall that in the Leibnizian framework, impulse derives from energy and inertia I derives from impulse. In this approach the **concept of inertia** I corresponds to the **constraint imposed on the Leibnizian structure to get only two conservation laws**. Formally, in the Einsteinian case, this corresponds to

$p = d_\mu E/dv_\mu = D_\mu\ dE/dv_\mu \ , \qquad\qquad I = E/c^2 = d_\mu p/dv_\mu = D_\mu\ dp/dv_\mu$

as shown in Eqs.(5,a) and (5,b). ***The fundamental difference between both rationalities – the "usual" (Lagrange-Hamilton) and the multiple (Huygens-Leibniz) one lies in the following fact: instead of deriving dE/dp starting from a Lagrangian on which one operates a Legendre transformation as well-known from analytical mechanics, dE/dp is deduced here from what we have called the "trans-subjective procedure" whose goal is to eliminate all points of view on motion keeping only the relation between conserved quantities (energy E and impulse p).*** This is done by considering the ratio between the two above given relations associated with impulse p and inertia I. Thus the infinite multiplicity of points of view given by $D_\mu = (E/E_0)^{a-\mu} = (E/mc^2)^{a-\mu}$ [see Eq.(8)] is eliminated by compensation leaving only a relation between conserved quantities. One may say that the Legendre transformation is to v = dE/dp in the "usual rationality" what the trans-subjective procedure is to this same relation in the "multiple rationality". The main difference lies in the fact that the Legendre transformation operates on the unique point of view associated with the ratio of space over time, eliminating it in favour of the impulse which is a conserved quantity, while the "trans-subjective" procedure operates on an infinite number of points of view at the same time eliminating them also in favour of the same conserved quantity. If in both cases one gets the same result, one should keep in mind that, ***in dynamics, the Leibnizian multiple rationality is subtler and more complete than the Lagrange-Hamilton formalism, since it includes it as a singular point of view among others*** as shown in the previous sections.

The singularity of this point of view is a direct consequence of the fact that the velocity concept, which turns out to be linked to the point of view of order 4, is the only one that satisfies simultaneously both conditions: $p = d_4E/dv_4 = D_4dE/dv_4$ and $v_4 = dE/dp$ as it may be verified from (8),(9),(15) and (16).

Let us finally note that the "Legendre transformation" as well-as the "trans-subjective procedure" are universal in the sense that they are independent of any specific dynamics. Here the attention has been focussed on Einstein's dynamics but the two different procedures apply also to Newton's dynamics where inertia $I \equiv M = m$ is constant in this case instead of being related to energy through $I \equiv M = E/c^2$.

I-3. Search for an "ontological order" hidden behind the "epistemological disorder".

With the advent of relativity theory, a number of controversies appeared. Lorentz-Poincaré interpretation (of length contraction considered to be a real physical effect) was replaced by Einstein's interpretation in terms of perspectives, adopted by most physicists. Another look at dynamics became possible with Minkowski, geometrizing relativity in a four dimensional space. This has been applied later on to electromagnetism. In this context, one distinguishes between the geometrical and dynamical contributions, related to one another through the concepts of duality and gauge invariance. Although the electromagnetic theory is distinct from the realm of dynamics, historically, electromagnetism was at the basis of the discovery of relativity theory. This historical relation between relativity and electromagnetism led to some problems of interpretation, the most famous one being related to the velocity of light that has no place in a purely dynamical framework. If this problem is better understood today [10-15] and [19-21], other problems concerning the concept of "relativistic mass" as well as that of incommensurability of physical theories are still questions of debate (as shown at length by Jammer [18]). However, if Leibniz seems far from all these debates on modern dynamics, this natural philosopher focussed the attention, at various occasions, on the danger of neglecting the necessary constraints in favour of superficial ones. In addition, according to Leibniz, **if mathematical properties play a major role in physics these should not have the priority on the physical principles such as the principle of relativity**. The critics he addressed to the dynamical theories of his epoch defended by Descartes, Huygens, Newton and others show to be useful even today as shown in this work. In particular, we shall show, in addition to his idea of discrete points of view on motion that allows the discovery of what may be called an "ontological order" hidden behind the current "epistemological disorder" (left by the different partial and incomplete contributions), his idea of discrete possible worlds will also be helpful for a better understanding of the structure of dynamics. **If one admits that one of the tasks of science consists in obtaining a maximum of information through a unified framework carrying its "raison d'être", then the Leibnizian framework is well-adapted to this ideal**.

On combining the different results associated with the parabolic and hyperbolic forms to which an infinite number of perspectives is associated, one obtains a doubly infinite number of differential equations, associated with points of view and possible worlds denoted respectively by Greek and Latin indices μ and i. With regard to the **possible worlds**, let us recall that, in the 17^{th} century, Leibniz used to make a net distinction between a finite and an infinite development in series of a function that he used to call "algebraic expressions" and "transcendent expressions". Such a distinction is adapted to the present approach since effectively the Newtonian parabolic world ($y = a\, x^2/2 + b$ or $E = p^2/2m + E_0$) and the Einsteinian hyperbolic one ($y = d\, [1 + x^2]^{1/2}$

or $E = mc^2[\ 1 + p^2/m^2c^2]^{1/2}$) correspond precisely to algebraic and transcendent expressions. More precisely, the possible worlds may be cast in the following form: $M_i = m\ (E/E_0)^{(i-A)\ /\ (T-A)}$ so that for $i = A$ (Algebraic), one gets the Newtonian framework $M_A = m$ and for $i = T$ (Transcendent) one is left with the Einsteinian one $M_T = m(E/E_0)= E/c^2$. In order to simplify the mathematical expressions, let us choose $A = 0$ and $T = 1$ so that the form: $M_i = m(E/E_0)^{(i-A)\ /\ (T-A)}$ reduces to $M_i = m(E/E_0)^i$. The addition of this new discrete multiplicity leads to a doubly multiple ordered structure: possible worlds or dynamics accounted for through the Latin index i and multiplicity of points of view on motion given through the Greek index μ. One obtains the following relations:

$$\mathbf{M_i = d_{\mu i}{}^2E/dv_{\mu i}{}^2 = \{(E/E_0)^{(a-\mu)i}\, d/dv_{\mu i}\, [(E/E_0)^{(a-\mu)i}dE/dv_{\mu i}]\} = m(E/E_0)^i} \qquad \mathbf{(18)}$$

This equation constitutes a compact form including the two possible dynamically admissible worlds (i = 0, 1) as well as the two conventional rationalities: the "usual" and the "emergent" one.
In the world of order one (i = 1) where both rationalities are qualitatively and quantitatively distinguished, the "emergent rationality" corresponds to the point of view of order: μ = a while the "usual rationality" is associated with the point of view of order: μ = a + 2. (Let us recall that Eq.(18) does not correspond to the most general case dealt with later on)

Among the possible dynamics (Leibnizian possible worlds) the two admissible ones which are associated with Newton's and Einstein's dynamics correspond to

$$i = 0 \text{ (Newton)} \qquad i = 1 \text{ (Einstein)} \qquad (19)$$

The doubly infinite (i, μ) second order differential system of equations given in (18) may be decomposed into two first order systems as follows:

$$d_{\mu i}p/dv_{\mu i} = m(E/E_0)^i \quad , \qquad p = d_{\mu i}E/dv_{\mu i} \qquad (20, a)$$

with

$$d_{\mu i}/dv_{\mu i} = D_{\mu i}\, d/dv_{\mu i} = (E/E_0)^{(a-\mu)i}\, d/dv_{\mu i} \qquad (20, b)$$

As to its "trans-subjective" version, it may be written as

$$d_{\mu i}{}^2E/dv_{\mu i}{}^2 = p\ dp/dE = m(E/E_0)^i \qquad (21, a)$$

from which one deduces the following integral form

$$p^2 = 2\ m\ E_0\ [(E/E_0)^{i+1} - 1]/(i+1) \qquad (21, b)$$

One recognizes here both Newtonian and Einsteinian dynamics given separately so that Newton's dynamics is not looked at only as a local approach, included in the Einsteinian one: it has its internal logic and it corresponds to the solution associated with the "algebraic expressions" while Einstein's solution corresponds to the "transcendent expressions" (Leibniz denominations). In geometrical terms one may say that the Newtonian dynamics is to Euclidean geometry what Einstein's one is to hyperbolic geometry. These two possible interpretations of Newtonian dynamics in its relation to Einstein's one (local versus global) is still a question of debate, as developed at length by Jammer in Ref.[18]. Both sides of the controversy can be equally well defended, so that at this point philosophical considerations come into play. In scientific circles, there is a tendency that consists in believing that such philosophical considerations are of no relevance to positive science. This is not true, since the research in one direction or another depends on the adopted interpretation of dynamics. If one admits that Einstein's dynamics is the last word corresponding to the final solution, then any new research on dynamics would be rejected, since the problem is considered to have been solved once for all. No one doubts today that the earth is a globe and not a plane or a disc; the same holds for dynamics, of which hyperbolic geometry constitutes the very heart. However, if one adopts another line of thought, where the attention is drawn on the idea of scale, then one can consider that Newton's dynamics is valid at some scale, Einstein's one at another wider scale, so that the horizon remains open to new discoveries where new elements may enter into the dynamical picture at some yet unknown scales. Although the analogy associated with the passage from a flat to a round earth is tempting, one should recognize that unlike dynamics, the form of the earth is finite and its contour may be delimitated. This does not hold for dynamics, since we are still far from the high energies needed to verify it on dynamical phenomena that occur at energy scales unreachable through presently available methods of measurements. If some concrete analogies may be well adapted for a better understanding of some abstract physical ideas, one should also look for the elements that show the limits of such an analogy. If analogy breaks down at a certain horizon then it is only partial and may become dangerous leading to false conclusions and fallacious statements.

Leibnizian distinctions: From possible worlds to the actual one through compossibles.

One of the main points that deserve to be mentioned here concerns the infinite multiplicity of inclusive points of view associated with an infinite multiplicity of exclusive possible worlds from which one should select the best. Contrary to Spinoza and according to Leibniz, this selection should be done in two steps, using different kinds of principles associated respectively with the intellect for the first kind, and with the will for the second (only the first is retained by Spinoza who influenced Einstein). The first kind allows passing from possible worlds to what Leibniz calls compossible ones. This passage is governed by a **logical necessity** associated here with the **principle of dynamical relativity**. The second kind that allows passing from the compossible worlds to the actual one (called also the best) is governed by moral necessity (Leibnizian denomination) intimately related to structural richness. This Leibnizian way of thinking is fully realised here. The system of equations given in (18) contains effectively two different kinds of infinities: an exclusive and an inclusive one, accounted for through the Greek and Latin indices and corresponding respectively to possible worlds and points of view on each world. The application of the "dynamical relativity principle" on the infinite number of possible worlds leads to two compossible ones: only $i = 0, 1$ are "dynamically admissible". For Leibniz, the passage from the two compossible worlds to the actual one is obtained neither by the recourse to experiment as usually done in physics, nor by a recourse to some logical principle. Here the selection criterion depends on the will that chooses the structurally richest world. A simple substitution in (18) of the values associated with the two compossible worlds leads to

$$M_1 = d_{\mu 1}{}^2E/dv_{\mu 1}{}^2 = \{(E/E_0)^{(a-\mu)} d/dv_{\mu 1}[(E/E_0)^{(a-\mu)}dE/dv_{\mu 1}]\} = m(E/E_0) \quad \text{for } i=1 \qquad (22)$$

$$M_0 = d_{\mu o}{}^2E/dv_{\mu o}{}^2 = d/dv_{\mu o}[dE/dv_{\mu o}] = m \qquad \text{for } i = 0 \qquad (23)$$

One easily shows that the world of order $i=1$ is structurally much richer than that of order $i = 0$. In particular, the latter turns out to be structurally equivalent to anyone of the different degenerate worlds. Indeed, on assuming the following local condition: $E \rightarrow E_0, \forall\, i \neq 0$, one gets from (18)

$$d_{\mu i}{}^2E/dv_{\mu i}{}^2 = d/dv_{\mu i}[dE/dv_{\mu i}] = m \qquad \text{for } E \rightarrow E_0, \forall\, i \qquad (24)$$

Drawing the curves associated with the solutions of the system of Eqs. (18) shows that this corresponds geometrically to a tree like structure, where Eq. (24) is confined to the trunk where all the curves coincide. The idea of structural richness associated with the Leibnizian framework is somehow qualitative and much more convincing than the one associated with the distinction between two curves (for example parabolic curve and hyperbolic or elliptic ones). Here the distinction is made between one curve [parabolic because of the solution of (23)] and a tree like structure composed of an infinite number of curves which only tend locally to the parabolic curve since all the solutions coincide in the vicinity of the origin forming thus a unique trunk of the tree like structure.

This clearly shows the fundamental difference distinguishing a Leibnizian dynamical approach from all presently available ones each of which dealing with only one curve that generalizes the parabolic Newtonian one.

Link to pre-Newtonian dynamical frameworks.

Before the end of the first part of this work, it is worth establishing a certain link between the above discussion and 17^{th} century dynamics, since the expressions of Huygens and Descartes dynamics written in a unified framework, lead to the following form: $p^2 = a_i\, E^{i+1}$ with $i=0$ for Huygens and $i = 1$ for Descartes. One discovers here three elements in favour of the Leibnizian methodology. Firstly, one shows that the idea of multiplicity of possible worlds (here possible dynamics) is not only metaphysical and logical but also encountered in the realm of dynamics. Secondly, it turns out that among such an infinite multiplicity, only the two dynamics proposed by Descartes and Huygens are dynamically admissible since they verify (4, c). Thirdly, their structures are too close to Newtonian and Einsteinian dynamics especially when they are cast in a differential form. Theses considerations will be developed more extensively in different manners in the third part of this work. In particular, Descartes dynamics which corresponds to

$$p^2 = a_1E^2 \text{ with } a_1 > 0 \text{ and } E > 0 \iff E = c\,|p| \quad \text{with } a_1c^2 = 1 \qquad \text{for } i = 1 \qquad (25)$$

will be regularized and connected to “doubly special relativity” where its structure turns out to be valid at some energy scale noted by E_M (as shown in the second part of this work and especially in Appendix L where extensive developments are performed).

Let us end this first part by quoting Michel Serres [28] as to the distinction he attributes to Leibniz concerning the realm of necessity and the realm of harmony. In pp. 487 and 488 of Ref.[28], he writes that ***determination is to necessity what harmony is to completeness. On the one hand, Leibniz looks for the necessary and sufficient conditions while on the other hand he exposes a unified totality. Leibniz looks for the minimal number of hypotheses associable with a maximum of information. An element may be redundant for necessity while indispensable for harmony that relates the “one” with the “many” in a unified framework. In Leibniz’s approach necessity seems to be of an axiomatic nature while harmony is of a structural one. The main Leibnizian question consists in the inquiry on the minimal necessary conditions in view of a maximal totalized harmony***.

These last considerations played a major role in my quest for a possible construction of a Leibnizian dynamical framework where both efficiency and intelligibility go hand in hand supporting each other for a better grasp of the relativity principle in its most general form.

SECOND PART

Summary: different extensions: (1) finiteness, (2) broken parity and (3) discontinuity.
The second part of this work concerns three different extensions. The first is linked to a previously misunderstood work, to which the present methodology provides a physical justification in direct relation with the relativity principle. This work turns out to be intimately associated with the framework of "doubly special relativity", or "deformed special relativity", accounted for in high energy physics. It is also shown that Descartes dynamics possesses an unsuspected aspect that may be directly linked to modern physics. The second extension deals with the dynamical relativity principle, where the parity requirement (usually associated with isotropy) is not imposed anymore. This leads to other solutions than those provided by conventional approaches. The third extension shows the structural richness of the discontinuous Cartesian solutions whose physical interpretation is directly linked to the historical problem of the cohesion of substance, difficult to understand and impossible to deal with through the previous analytical formulations.

II-1. First extension: dynamics and electromagnetism.

a) Particle like interpretation (extended dynamics with a bounded energy).

I realized, twenty years ago, that both Einstein's dynamics: $E^2 - c^2p^2 = E_0^2$, with $E_0= mc^2$, and the Newtonian one: $E = p^2/2m + E_0$, satisfy a unique differential equation: $p'' + m^2/p^3 = 0$ or equivalently $p^3p'' = - m^2$ with the following notation $p' = dp/dE$. A simple derivation with respect to E allows to eliminate the remaining constant associated with the mass concept leading to $p\, p''' + 3\, p'\, p'' = 0$. [Notice that this third order differential equation is none other than the one given by Eq.(4,c)]. Thus, both dynamics are somehow unified, the difference lying in the treatment of limit conditions. I also realized that on starting from the same differential equation and choosing other limit conditions, it is possible to construct a dynamical framework where energy and impulse do not tend to infinity any more so that one gets a second coupling parameter that adds to the one usually associated with the "velocity of light" which couples space to time. Noting the well-known analogy between $E^2 - c^2p^2$ (dynamical relativity) and $\partial^2/\partial t^2 - c^2\, \partial^2/\partial x^2$ (electromagnetism or wave like structure) I showed that the extension of dynamics leads to the extension of electromagnetism where a new coupling constant, associated with a (microscopic) length enters into play. (When this length vanishes, one recovers the classical theory). At this epoch I was not aware of the link between such generalizations and the relativity principle, so I could not justify their possible interest from a physical standpoint. However, in spite of the absence of a unique solution, one should notice that the above 3rd order differential equation imposes a strong constraint, so that one may single out three uncoupled solutions on a rational ground. (A summary of this work may be consulted in the (1987-1989) scientific report of LPMO-CNRS, pp.7-8. The basic elements of this report are translated and commented in Appendix N). I discovered recently that among the three different uncoupled solutions two of them correspond to recent dynamical approaches as shown in the forthcoming developments. Indeed, on looking for a generalization of $p^2 = E^2 - E_0^2$ ($c = 1$, natural units) so that energy and impulse do not tend simultaneously to infinity as usual but to some upper limit and accounting for the constraint given in (4, c), I obtained the following class of solutions

$$p^2 = [E^2 - E_0^2\, g_{kl}(E/E_M)/g_{kl}(E_0/E_M)] \qquad (26)$$

with

$$g_{kl}(E/E_M) = A_{kl} \,(1 - [E/E_M]^{\,k})^{l} \qquad (27)$$

and

$$\{k,l\}= [\{1,1\};\{1,2\}; \{2,1\}] \qquad (28)$$

It is readily checked out that when the upper limit E_M tends to infinity, one recovers Einstein's dynamics. In addition, when E tends to the upper limit E_M, then E^2 tends to p^2 so that the upper limit E_M plays exactly the same role as that played by infinity in Einstein's dynamics.

Relation to "doubly or deformed special relativity".
Among these three well-determined solutions two of them correspond exactly to two dynamics [7,8] recently developed independently in the framework of so-called "doubly or deformed special relativity". These two solutions correspond to the second and third couples $\{k,l\}= [\{1,2\}; \{2,1\}]$ given in (28).

These uncoupled solutions are obtained from the general coupled form expressed through the following development in series (see Appendix L for more details on the explicit solutions developed in relation to pre-Newtonian and post-Einsteinian dynamical frameworks also discussed in the last paragraph of this section entitled: "direct passage from pre-Newtonian to post-Einsteinian dynamics").

$$f(E) = \Sigma\, C_{kl}[(\Delta E^k)^l] \qquad \Delta E^k = E^k - E_M^{\,k} \qquad k, l \in N^+$$

This expression should satisfy $f\,'''(E) = 0$ to be compatible with the dynamical relativity principle. It is written in such a manner that it verifies the upper limit condition $f(E_M) = 0$. A simple transformation allows one to express these equations under the following form

$$f(E) = \Sigma\, C_{kl}[(\Delta E^k)^l] \;=\; \Sigma\, A_{kl} \,(1 - [E/E_M]^{\,k})^{l} \;= \Sigma\, g_{kl}(E/E_M)$$

The uncoupled solutions are obtained by letting the different coefficients of the series A_{kl} vanish except one. This procedure is repeated until one takes into account all the available possibilities. The solutions are given explicitly in Appendix L.

b) Wave interpretation (extension of Klein-Gordon equation).
The wave equation corresponding to Eq.(26) takes on the following form:

$$[\partial^2/\partial x^2]\, \psi(x,t) = [\partial^2/\partial t^2 + k_{0mkl}^{\;2}\, g_{kl}\,(i\, x_m\, \partial/\partial t)\,]\, \psi(x,t) \qquad (29)$$

where we have set

$$k_{0mkl}^{\;2} = k_0^{\,2}/\, g_{kl}\, (k_0\, x_m) \qquad (30)$$

The passage from a particle like interpretation to a wave like representation consists in operating the usual correspondence relations

$$E\rightarrow i\,\partial/\partial t \qquad p\rightarrow -\,i\,\partial/\partial x \;\Leftrightarrow\; E/E_M \rightarrow i\,x_m\,\partial/\partial t \qquad p/E_M \rightarrow -\,i\,x_m\,\partial/\partial x \tag{31 }$$

where we recall that in natural units, one has $E_M\,t_m = E_M\,x_m = 1$, since energy and time are inversely proportional, while space and time become equivalent. Thus, a maximal energy corresponds directly to a minimal length. As to rest energy, it may also be accounted for through frequency or wave number such that $E_0 = \omega_0 = k_0$. With these considerations one gets the following equivalent relations

$$E_0\,/E_M = k_0\,x_m \qquad \text{or} \qquad E_0\,/E_M = \omega_0\,t_m \tag{32}$$

so that the passage from a particle like picture to a wave like picture corresponds to the following replacements

$$g_{kl}(E/E_M)\rightarrow g_{kl}\,(\,i\,x_m\,\partial/\partial t\,) \qquad g_{kl}\,(E_0/E_M)\rightarrow g_{kl}\,(k_0\,x_m) \tag{33}$$

For $x_m \rightarrow 0$, one is left with the well-known wave equation

$$[\partial^2/\partial x^2]\,\psi(x,t) = [\partial^2/\partial t^2 + k_0{}^2]\,\psi(x,t) \tag{34}$$

corresponding to the Klein-Gordon equation.

(This work being confined to one dimensional physics as mentioned earlier, one may refer to my previous work Ref.[26] (pp.187-193) where in chapter 8 devoted to electromagnetic interactions in 4-dimensional and (3+1)-dimensional frameworks, rational correlations are made between relativity, electromagnetism and some other equations among which those of Klein-Gordon and Schrödinger).

Although the solutions given by Eqs.(26) and (29) were obtained long ago (after having discovered that both Newtonian and Einsteinian dynamics may be cast into a unified framework), no physical interpretation was proposed. The constraint (4, c) which singles out only three possible uncoupled solutions (related to the relativity principle) was not well understood. It was purely structural. In my (above-mentioned) report, I explained that the presented approach treats the problem from a dynamical point of view with **no emphasis on space and time** so that the principle of relativity appeared not to play a major role in the process of generalization. At this period I was not conscious of the possibility of dealing directly in a dynamical way with the principle of relativity (without any specification on space and time). I understood this later on, thanks to the works of J.M. Lévy-Leblond and C. Comte. Consequently, the mathematical solutions, **exactly similar** to the ones reproduced here, since they use the same methodology and the same third order differential equation (4, c), lacked an appropriate interpretation. Contrary to what was believed before, **the constraint at the basis of the coupled and uncoupled solutions is none other than the manifestation of the dynamical relativity principle**. This explains the reason why such solutions could be considered to be simple mathematical generalizations.

Transposition of what results from the unbounded energy to the bounded one.

Although the transposition from an unbounded framework to a bounded one cannot be done globally in the general case, it turns out that the different extended solutions deduced here, possess such a singular form that a global transposition turns out to be possible. One may recover a hyperbolic form by performing the following transformations

$$D = [g_{kl}(E/E_M)]^{-1/2}\, E \qquad D_0 = [g_{kl}(E_0/E_M)]^{-1/2}\, E_0 \qquad q = [g_{kl}(E/E_M)]^{-1/2}\, p$$

leading to $D^2 - q^2 = D_0^2$. This form, similar to the fundamental equation of Einstein's dynamics: $E^2 - p^2 = E_0$ (in natural units) suggests to transpose all what has been developed for the couple (E, p) to the couple (D, q). In particular, the compact second order system of equations given in (18), including the "usual" and "emergent" rationalities (two singular points of view among others), is assumed to apply in the present extended framework by simply operating the following correspondences: $E \to D$, $E_0 \to D_0$ and $p \to q$. All other considerations remain invariant, they allow passing directly from the conventional "unbounded" world ($E \to \infty$) to the "bounded" one ($E \to E_M$). This is a direct consequence of the global character associated with the above particular form where the ratios: D/E and q/p are equal to one another as shown in the last equations. This applies because the bounded world with respect to E is equivalent to an unbounded one with respect to D, since one has: $D \to \infty \Leftrightarrow E \to E_M$. Obviously, these formal considerations and structural correspondences require a more physical interpretation as to the significance of the couple (D, q) and its relation to conservation laws. A number of questions are open to discussion at this level.

Direct passage from pre-Newtonian to post-Einsteinian dynamics.

It is worth noting that the three different solutions are also deducible from a regularization of Descartes dynamics. If one starts from $p^2 = a\,E^2$, where energy (positive entity) is proportional to the absolute value of impulse $|p|$ (irregularity or non-analyticity), and requires that this form be valid only locally at some energy scale E_M then one may look for solutions having the following regularized form: $p^2 = a\,E^2 + \kappa f(E)$, such that $f(E) \to 0$ when $E \to E_M$ so that one recovers Descartes dynamics at this scale.

To this end, one looks for a general analytical solution (developable in series) as follows:
$f(E) = \Sigma\, C_{kl}[(E^k - E_M{}^k)^l]$, $k, l \in N^+$ which automatically satisfies $f(E_M) = 0$. Because of the constraint imposed by the dynamical relativity principle, only the values given in (28) are admissible. (One may refer to Appendix L for details). The specificity of such a Leibnizian methodology is to eliminate only **what is rationally inadmissible** (Descartes' dynamics was rejected in the 17th century on an empirical ground). For Leibniz, experimental evidence is local by essence, so that one form may be inadmissible at one scale but admissible at another one. This is not only true for Newton's dynamics, but also for Descartes one even if the irregular character of this dynamics and the too narrow physical methods limited to regular functions prevented the physicists from deducing such a statement. **It is remarkable to note that the Leibnizian extension of Descartes dynamics does not only lead to Einstein's one but to a more general form since Einstein's dynamics is recovered only when the scale E_M at which Descartes dynamics is valid is cast to infinity $E_M \to \infty$.** (See Appendix L for details).

The basic reason for which the extension of Descartes dynamics leads to a structurally richer solution than the extension of Newton's one is intimately related to the more general framework in which Descartes dynamics is embedded, since it belongs to the class of **generalized functions**. In a mathematical sense, one may say that Descartes dynamics is potentially richer than the Newtonian analytical framework, since it contains potential information unreachable through analytical methods. One should add that Newton's dynamics is not only local with respect to Einstein's dynamics $\{E = E_0 [1 + (cp/E_0)^2]^{1/2}\}$ but also with respect to any even function $\{E = E_0 f(cp/E_0) = E_0 f(-cp/E_0)\}$: any dynamics would lead locally to a dynamical framework of the Newtonian type. This clearly shows that a regularization of Descartes dynamics leads automatically to a structure that includes the Newtonian one. In spite of its local character, emphasized by Leibniz, (corresponding to its discrepancy in the vicinity of the origin), Descartes dynamics contains more potentialities than those provided by the Newtonian framework. *Another explicit manifestation of this fact will be given in the third extension, associated with the discontinuity of Descartes dynamics, where the Cartesian structure will again include not only Newton's dynamics but also Einstein's one. In the above example, the lack of analyticity is hidden and does not appear explicitly, since one deals with* $p^2 = a E^2$ *and its derivatives with respect to E instead of the equivalent form* $E = c\,|p|$ *whose derivative with respect to p (v = dE/dp, first canonical Hamilton equation) is discontinuous as it will be explicitly discussed at the end of this work through the application of Lagrange-Hamilton formalism to Descartes dynamics.*

II-2. Second extension: Solution of the general case with broken parity.

We start by the resolution of the point of view associated with the additive composition law, before dealing with the three other points of view developed in the first part of this work. Then, the solution is extended to finite energy, following the line of thought developed in the previous section (and in Appendix L where the extension of Descartes dynamics is explicitly developed).

a) Solution associated with the additive point of view on motion.

In order to underline the degrees of freedom provided by the principle of relativity when the latter is not constrained by the parity requirement (isotropy), let us consider the general form of the relativity principle given in (1), (2) and (3). To this end, we shall restrict ourselves to the basic additive point of view that we note by w (p = dE/dw), so that the generalized derivative ($d_\mu/dv_\mu = D_\mu\, d/dv_\mu$) coincides with the usual one (d/dw) leading to $d^2E/dw^2 = \lambda E + \gamma dE/dw + \eta$. In addition, we let the constant η vanish obtaining thus, $d^2E/dw^2 = \lambda E + \gamma dE/dw$. On replacing E by $R = E + \eta/\lambda$ one gets the following second order equation: $d^2R/dw^2 = \lambda R + \gamma dR/dw$ which is equivalent to the one expressed with E subject to $\eta = 0$. We aim here at examining the influence of parity breaking due to $\gamma \neq 0$. Thus, one obtains a more general solution than that of the "emergent rationality" where isotropy is postulated from the start ($\gamma = 0$). We shall not present here the calculations. We underline simply the result which may be cast in the following particularly interesting form:

$$e = E_0 \exp(aw) \cosh(dw) \quad (35, a)$$

$$P = d\, E_0 \exp(aw) \sinh(dw) \quad (35, b)$$

where we have set

$$d = (a^2 + b^2)^{1/2} \qquad b^2 = \lambda \qquad a = \gamma / 2 \qquad (35, c)$$

and where the entities e and P turn out to be related to E and p through the following combinations

$$P = (d^2/b^2)\, p \qquad e = E + (a / b^2)\, p \qquad (35, d)$$

We recall that if E and p correspond to conservation laws, then e and P lead also to two equivalent conservation laws. In addition, one may easily notice that P is linked to e in the following way (instead of the usual derivative)

$$P = \exp(aw)\, d/dw\, [\, \exp(–aw)\, e] \qquad (35, e)$$

Obviously, on applying this new "generalized derivative" to P, one is led to an expression having the dimension of a mass and related to e through a simple proportionality relation. Thus, the equivalence between inertia or mass and energy is recovered except that the equality between e(w) and e(–w) is lost because of anisotropy.

b) Expression of the solution according to the different points of view.

In this part, an extension is proposed to account for the different points of view. To this end, we use the results given in Eqs.(15)-(16) associated with the hyperbolic structure. Thus, one sets

$$\hat{u}_2 = dw = [a^2 + b^2]^{1/2}\, w = [(\gamma/2)^2 + \lambda]^{1/2}\, w \qquad (36, a)$$

and accounts for the four different points of view already dealt with previously. One is then left with the following expressions

$$e/ E_0 = [\exp\hat{u}_2]^{a/d} \cosh(\hat{u}_2) = [\, \{1 + \sin(\hat{u}_3)\} \sec(\hat{u}_3)]^{a/d} \sec(\hat{u}_3) \qquad (36, b)$$

$$P/ E_0 = d\, [\exp(\hat{u}_2)]^{a/d} \sinh(\hat{u}_2) = d\, [\, \{1 + \sin(\hat{u}_3)\} \sec(\hat{u}_3)]^{a/d} \tan(\hat{u}_3) \qquad (36, c)$$

For the couple of points of view of order (2, 3) and

$$e/ E_0 = [\, (1 + \hat{u}_1^2)^{1/2} + \hat{u}_1]^{a/d} [1 + \hat{u}_1^2]^{1/2} = \{[(1 + \hat{u}_4)/ (1 – \hat{u}_4)]^{a/2d} \}/ [1 – \hat{u}_4^2]^{1/2} \qquad (36, d)$$

$$P/ E_0 = d\, [\, (1 + \hat{u}_1^2)^{1/2} + \hat{u}_1]^{a/d}\; \hat{u}_1 = d\{[(1 + \hat{u}_4)/ (1 – \hat{u}_4)]^{a/2d} \}\{\, \hat{u}_4/ [1 – \hat{u}_4^2]^{1/2}\} \qquad (36, e)$$

for the other couple (1, 4). These expressions are given in such a way that when the anisotropic coefficient vanishes (a = 0), one recovers the usual solutions associated with Eqs.(15)-(16).

c) Extension to finite approaches linked to "doubly or deformed special relativity".

We have seen earlier that the passage from infinite to finite dynamical frameworks is easy to obtain since a global transformation is at work allowing directly to get solutions for which energy and impulse do not tend to infinity as usually the case for Newton and Einstein dynamics. What has been applied in the isotropic case is extended to the anisotropic one (broken parity). To this end, one should replace E_0 , e and P respectively by

$$E_{0kl} = E_0 / [1 - (E_0/E_M)^k]^{l/2} \quad e_{kl} = e / [1 - (e/E_M)^k]^{l/2} \quad P_{kl} = P / [1 - (e/E_M)^k]^{l/2} \tag{37, a}$$

as shown in Appendix L as well as in the first extension dealing with the passage from infinite to finite approaches. In particular, when energy $E_M \to \infty$, E_{0kl}, e_{kl} and P_{kl} reduce to E_0 , e and P respectively. The transformation is immediate so that the above equations take on the following forms

$$e_{kl} / E_{0kl} = [\exp(\hat{u}_{2kl})]^{a/d} \cosh(\hat{u}_{2kl}) = [\{1 + \sin(\hat{u}_{3kl})\} \sec(\hat{u}_{3kl})]^{a/d} \sec(\hat{u}_{3kl}) \tag{37, b}$$

$$P_{kl} / E_{0kl} = d [\exp(\hat{u}_{2kl})]^{a/d} \sinh(\hat{u}_{2kl}) = d [\{1 + \sin(\hat{u}_{3kl})\} \sec(\hat{u}_{3kl})]^{a/d} \tan(\hat{u}_{3kl}) \tag{37, c}$$

for the couple of points of view of order (2, 3) and

$$e_{kl} / E_{0kl} = [(1 + \hat{u}_{1kl}^2)^{1/2} + \hat{u}_{1kl}]^{a/d} [1 + \hat{u}_{1kl}^2]^{1/2} = \{[(1 + \hat{u}_{4kl})/ (1 - \hat{u}_{4kl})]^{a/2d} \}/ [1 - \hat{u}_{4kl}^2]^{1/2} \tag{37, d}$$

$$P_{kl} / E_{0kl} = d[(1 + \hat{u}_{1kl}^2)^{1/2} + \hat{u}_{1kl}]^{a/d} \hat{u}_{1kl} = d\{[(1 + \hat{u}_{4kl})/ (1 - \hat{u}_{4kl})]^{a/2d} \}\{ \hat{u}_{4kl}/ [1 - \hat{u}_{4kl}^2]^{1/2}\} \tag{37, e}$$

for the other couple (1, 4).

The four different points of view $\hat{u}_\mu$ with (μ = 1, 2, 3, 4) have been replaced by $\hat{u}_{\mu kl}$. Contrary to the Greek index μ that accounts for points of view, the Latin indices k and l account for the three different solutions (possible worlds) deduced from the application of the relativity principle to dynamical frameworks where an additional coupling constant occurs through energy. In particular, when energy $E_M \to \infty$, one recovers the expressions of e and P given in Eqs.(36). Apart from these intermediate results given in the above paragraphs (b) and (c), let us go back to the starting point of this section dealing only with the rapidity parameter and develop it for two of the three solutions which verify kl = 2. Indeed, if one singles out the two solutions associated with the couples (k, l) = {(1, 2); (2, 1)} one notices the following remarkable property kl = 2 so that only one index is needed in this restricted situation. Thus, Eqs.(37, a) may be cast in the following form

$$E_{0k} = E_0 / [1 - (E_0/E_M)^k]^{l/k} \quad e_k = e / [1 - (e/E_M)^k]^{l/k} \quad P_k = P / [1 - (e/E_M)^k]^{l/k} \quad k = 1, 2 \tag{38}$$

This transforms (37, b) and (37, c) as follows:

$$e_k = E_{0k} [\exp(\hat{u}_{2k})]^{a/d} \cosh(\hat{u}_{2k}) \tag{39, a}$$

$$P_k = E_{0k} d [\exp(\hat{u}_{2k})]^{a/d} \sinh(\hat{u}_{2k}) \tag{39, b}$$

The substitution of (38) into (39, a) and (39, b) allows one to write the following explicit expressions

$$e = E_{0k} \{[\exp(\hat{u}_{2k})]^{a/d} \cosh(\hat{u}_{2k})\}/\{1 + [E_0/E_M [\exp(\hat{u}_{2k})]^{a/d} \cosh(\hat{u}_{2k})]^k\}^{1/k} \tag{39, c}$$

$$P = E_{0k}\, d \{[\exp(\hat{u}_{2k})]^{a/d} \sinh(\hat{u}_{2k})\}/\{1 + [E_0/E_M [\exp(\hat{u}_{2k})]^{a/d} \cosh(\hat{u}_{2k})]^k\}^{1/k} \tag{39, d}$$

Particular and limit cases: **$\gamma = 0$ (parity or isotropy) and $E_M \to \infty$ (unbounded energy)**

In the isotropic case ($\gamma = 0$) or equivalently ($a = 0$), the above expressions much simplify leading to

$$e = E = E_{0k} \{ \cosh(\hat{u}_{2k})\}/\{1 + [E_0/E_M \cosh(\hat{u}_{2k})]^k\}^{1/k} \tag{40, a}$$

$$P = p = E_{0k}\, b \{ \sinh(\hat{u}_{2k})\}/\{1 + [E_0/E_M \cosh(\hat{u}_{2k})]^k\}^{1/k} \tag{40, b}$$

where $d = [a^2 + b^2]^{1/2}$, and $[\exp(\hat{u}_{2k})]^{a/d}$ reduce to b and unity respectively. For $e = E$ and $P = p$, one may refer to (35,d) to verify this reduction.

In the case of an unbounded energy ($E_M \to \infty$) the two solutions $k = 1$ and 2, merge into a unique solution and one is left with the usual Einsteinian dynamics associated with the “emergent rationality” expressed through rapidity w given by

$$E = E_0 \cosh(bw) \tag{40, c}$$

$$p = b\, E_0 \sinh(bw) \tag{40, d}$$

The distinction between $\hat{u}_{21}$ and $\hat{u}_{22}$ disappears ($\hat{u}_{21} = \hat{u}_{22} = w$) since the two dynamics coincide at this scale.

II-3. Third extension: discontinuity in the history of dynamics.

In the 17^{th} century, Leibniz did not criticize Descartes dynamics only by asserting that Descartes failed to propose a continuous passage from rest to motion, but he also asserts that Descartes dynamics does not respect the cohesion of substance. How can this be interpreted in the light of Leibniz’s conception of substance? One should firstly recall that before the advent of Newtonian mechanics associating substance – a positive constant – with the mass concept, in pre-Newtonian dynamics one may still distinguish between active and passive positive substances. According to Leibniz, the passive substance corresponds to active substance from which activity due to motion was excluded. In modern terms, this corresponds to energy in motion and at rest relatively to some reference frame. If one tries to understand the meaning of Leibniz’s assertion by use of the usual Newtonian approach dealing with Descartes dynamics ($Q = m|u|$, $p = mu$ where Q is proportional to energy $Q = bE$), then one encounters a serious difficulty of interpretation. However, If one looks at the same problem in the light of the present Leibnizian approach, things become clear and the obtained discontinuous solutions turn out to be interesting, leading to nontrivial results. Indeed, if one deals with Descartes dynamics ($|p| = bE \Leftrightarrow p_\pm = \pm b E_\pm$, $E_\pm > 0$, $b>0$) following the line of thought developed by Leibniz and adopting the point of view associated with rapidity w ($p = dE/dw$) on either side of the discontinuity, then one is led to the following result

$$E_{\pm} = E_{0\pm} \exp(\pm b w_{\pm}) \qquad w_{+} > 0 \ , \ w_{-} < 0 \qquad b > 0 \tag{41}$$

$$p_{\pm} = \pm\, b\, E_{0\pm} \exp(\pm b w_{\pm}) \tag{42}$$

One immediately verifies that Descartes dynamics satisfies the dynamical relativity principle and the main problem concerns its discontinuity at the origin as emphasized by Leibniz. This dynamical framework may also be expressed in the following form

$$E = \exp(bz)[\, E_0 \cosh(bw) + D_0 \sinh(bw) \,] \tag{43}$$

$$p = b \exp(bz)[\, E_0 \sinh(bw) + D_0 \cosh(bw)] \tag{44}$$

if one accounts for the following mean values and differences

$$E_0 = \tfrac{1}{2}\, [E_{0+} + E_{0-}] \qquad D_0 = \tfrac{1}{2}\, [E_{0+} - E_{0-}] \tag{45,a}$$

$$E = \tfrac{1}{2}\, [E_{+} + E_{-}] \qquad p = \tfrac{1}{2}\, [p_{+} + p_{-}] \tag{45,b}$$

$$w = \tfrac{1}{2}\, [w_{+} + w_{-}] \qquad z = \tfrac{1}{2}\, [w_{+} - w_{-}] \tag{45,c}$$

At this point one easily establishes a direct articulation between the state of rest and that of the cohesion of passive substance. Indeed, if one assumes the following constraint

$$w = 0 \ \Leftrightarrow p = 0 \tag{46}$$

then and only then one is led to

$$D_0 = 0 \Leftrightarrow \ E_0 = E_{0+} = E_{0-} \tag{47}$$

so that one recovers the cohesion of passive substance and the above equations reduce to

$$E = E_0 \exp(bz) \cosh(bw) \tag{48,a}$$

$$p = b\, E_0 \exp(bz) \sinh(bw) \tag{48,b}$$

Some final remarks:
In spite of the structural analogy between (48,a)-(48,b) and (35,a)-(35,b) one should notice that in (48,a)-(48,b) one deals with two variables w, z and one constant b while in(35,a)-(35,b) one has two constants a, d and one variable w.

It is worth noting that on deriving twice the energy E with respect to motion w, one gets again a law equivalent to the energy E, in so far as the conservation properties are concerned which constitute another manifestation of the dynamical relativity principle. This could be expected without any calculation, since any linear combination remains compatible with this principle. Notice also that any derivation with respect to z keeps each quantity equivalent to itself.

It is remarkable to discover that contrary to what is believed, Descartes dynamics includes more potentialities than Newtonian dynamics in so far as conservation properties are concerned. If 17th century physicists had adopted the Leibnizian procedure to deal with the relativity principle they would have discovered a wealth of interesting properties impossible to obtain from the Newtonian degenerate dynamical framework.

THIRD PART

Summary.
In this part of this work we shall examine the problems met by the "usual" and "emergent" rationalities in dealing properly with Descartes dynamical framework. After revealing the reasons for which Descartes dynamics was discarded by both rationalities, it is shown that this dynamics does not only satisfy the dynamical relativity principle but it also reveals an equivalence between mass and energy similar to the one encountered in Einstein's dynamics. Such a fact could not be revealed by the "usual" or "emergent" rationalities, for they are limited by two different restrictions: adoption of a single point of view on motion, and consideration of a continuous and analytical framework (too narrow to deal with such an irregular dynamics). Finally, some comments are given in favour of the Leibnizian paradigm which combines both qualitative intelligible and quantitative efficient physics in a coherent way where explanation and exploration go hand in hand.

Before developing quantitatively these considerations, let us recall some of the basic elements associated with the conciliatory character of the Leibnizian methodology with its multiple perspectives. It will be at the basis of the proposed solution concerning the dynamical validity of Descartes dynamics.

III-1. Importance of scales and points of view in Leibniz's philosophy of nature.

Leibniz considered the parabolic (Newtonian) framework to be valid only locally, at some scale, needing to be extended and deepened. He was much impressed by the internal structure of a drop of a biological liquid that he observed through the Leeuwenhoek microscope. This experiment convinced him that science was still in its childhood, and that before rejecting any approach on experimental grounds, by use of specific measurements, limited by essence, one should construct a general framework capable of dealing with dynamics at different scales, with various kinds of measurements getting beyond what meets the eye. He asserted that one of the main aims of his infinitesimal calculus, accompanied by his combinatorial methodology, at the basis of his metaphysical relational approach of nature, is to provide a sufficiently subtle formal language, dealing with perspectives unreachable by the usual methods of his time. More precisely, he paid a special attention to various problems examined through different perspectives among which the "catenary's curve". This curve is intimately linked to the hyperbolic form dealt with in the Appendices E and H. One may consult Refs. [1-6] to get a deeper account of the Leibnizian philosophy of nature, and more particularly Refs. [1,2,4,5] concerning dynamics.

The conciliatory Leibnizian attitude asserts that a framework should be rejected only if it violates the basic principles; otherwise, it should be simply corrected and completed. One should recognize that Descartes dynamics may be valid at some scale that 17th century experiments could not detect. Thus, although Leibniz preferred Huygens dynamics because of its compatibility with the experiments available at his epoch as well as with the continuity and dynamical relativity principles, the one proposed by Descartes should be examined on a rational ground, and rejected only if it violates the dynamical relativity principle that constitutes the heart of dynamics. It is quite well-known that Descartes was in favour of the relativity principle, but this does not mean that his quantitative dynamics is in agreement with his qualitative conceptual framework. At different occasions, **Descartes proposed valid qualitative methods that he applied in an incorrect manner** leading to false conclusions. (The most famous example is the Cartesian empirical proof of the infinity of light velocity. He committed a logical error, since one cannot prove such infinity. One can only demonstrate its finiteness as shown later on by Roemer using a more appropriate experiment: the distance considered was sufficiently large to detect the huge velocity with respect to usual ones but nevertheless finite. Descartes procedure was theoretically valid but its application to an insufficiently large distance did not allow the measurement of any finite value. He, thus, wrongly deduced that the light velocity was infinite).

Unlike the rejection of the Newtonian paradigm by scholastic philosophers, Leibniz critics were based on logical arguments and mathematical discoveries that he made in his works on differential calculus. Two of these discoveries will be recalled here. Although Huygens parabolic solution seems to be perfectly well adapted to the principles as well to experiments, Leibniz insists on the fact that a parabola may be interpreted as a local form of any even function. In this regard one should mention that sometimes **Leibniz used to call the "vis viva" or "living force" (mu^2) by the name of "absolute force" focussing thus the attention on the following property f(u) = f(–u) as shown in Ref.[4]. This allows including Descartes dynamics compatible with this symmetry requirement: m|u| = m|–u| . This general consideration allows different approaches to be fused into a unified qualitative framework on which the dynamical relativity principle should operate according to different points of view and in a direct link with conservation properties.** These properties are considered as primary, and representative of the notion of active substance inherited from Aristotle's metaphysics**.** It should be emphasized that the correction of Descartes dynamics, in order to recover continuity, will lead automatically to some even regular function. Since all even functions lead locally to a parabolic form, then one discovers that the corrected Cartesian dynamics will include locally Huygens one (formally equivalent to that of Newton). Thus, without any specific calculation, the global properties associated with functions are sufficient to draw remarkable conclusions. Leibniz went further since he had drawn the attention at different occasions on the unity brought by a differential equation which may lead to different solutions [9]. These solutions may appear to be incompatible with each other, some of them being possibly regular, while the others are irregular, forming thus two different classes of solutions. The irregularity of the Cartesian world may be a simple consequence of its local validity at some specific scale. If this is the case, one should assume that Descartes dynamics would be valid at some particular scale and look for an extension of this dynamics, in order to construct a **global framework** where Descartes dynamics turns out to be a local imprint, revealing only one of its facets and hiding others that remain to be discovered. [Instead of dealing directly with this question in the main text, we have recalled an old work developed previously and which turns out to be very close in its spirit to the above discussion, except that it deals with modern physical approaches (Newton and Einstein dynamics as well as electromagnetism and correspondence relations used extensively since the advent of quantum mechanics). The above-mentioned global framework, rooted in Descartes dynamics is examined in Appendix L].

After these qualitative considerations, let us enter into the realm of quantity, showing the reason for which Descartes dynamics is rejected by Huygens method and Lagrange-Hamilton formalism and the correction to bring to be rehabilitated through a methodology of a Leibnizian type. The proposed methodology is based on hypotheses, weakened in two directions, widening the previous dynamical analytic frameworks and leading thus to a greater number of degrees of freedom including possible discontinuities. This wider framework corresponds to an extended form of the "principle of dynamical relativity".

III-2. Application to pre-Newtonian dynamics.

Examination of Descartes dynamics by use of Huygens method as well as Lagrange-Hamilton formalism.

After having shown the interest of the Leibnizian methodology for a better understanding of modern science let us briefly draw the attention on the consequences brought by the above construction on pre-Newtonian dynamics and particularly Descartes one whose non analyticity led to its rejection by both Huygens method and Lagrange-Hamilton formalism. To see this, let us recall Descartes dynamics and that of Huygens which correspond respectively to the two following couples of equations:

1st couple $\rightarrow$ ($E = cm|u|$, $p = mu$) 2nd couple$\rightarrow$ ($E = 1/2\ mu^2$, $p = mu$). (49)

(i) Rejection of Descartes dynamics by use of Huygens method.

The reason for which Descartes dynamics is to be rejected if one applies Huygens procedure ($p = dE/du$) is due to the fact that only the second couple ($E = 1/2\ mu^2$, $p = mu$) verifies:

$p = dE/du$. More precisely, if one applies Huygens procedure to $E = c\ m|u|$, then one gets:

$p = dE/du = mc$ for $u > 0$ and $- mc$ for $u < 0$ (discontinuity at the origin). This contradicts $p = mu$ and leads to the discontinuity emphasized by Leibniz in his critics of Cartesian dynamics and its discrepancy concerning the passage from the state of rest to that of motion. Thus, **if Descartes dynamics is to be revived, then the concept of motion associated with it should differ from that of Huygens.**

Comment: When a given approach is compatible with the dynamical relativity principle, it is said to be "dynamically admissible". When it enters in the mould of Lagrange-Hamilton formalism that applies in different physical contexts, then it is said to be "physically admissible". Let us note that even if Huygens approach appears to be both "dynamically and physically admissible" its physical admissibility is only accidental. Indeed, if one applies Huygens method, where u is interpreted as the velocity concept ($u = v$), then one should write: $p = dE/dv$. This writing is not compatible with the Hamilton-Lagrange formalism where one has $p = dL/dv$. Thus, Huygens confuses energy with the Lagrangian even if both quantities are identical in the parabolic framework of classical mechanics and in the absence of any potential contribution, where one has $E = L = ½\ mv^2$. **The only way to revive Huygens method or point of view on motion is to recognize that v is not to be associated with the velocity concept kinematically associated with dx/dt and dynamically with $v = dE/dp$.**

(ii) Rejection of Descartes dynamics by use of Lagrange-Hamilton formalism.

In the light of the "usual rationality", the rejection of Descartes dynamics will be accounted for in two different ways.

1) First way: interpretation of u as a velocity.

If one interprets the Cartesian parameter u associating it with the velocity concept given through the first canonical Hamilton equation for free particles **(u = v = dE/dp)**, then the Cartesian couple reduces to

$$(E = cm|u|,\ p = mu) \Leftrightarrow (E = cm|dE/dp|,\ p = m\ dE/dp) \qquad (50)$$

It is immediately verified that the two relations are contradictory.

2) Second way: discontinuity and impossibility of forming an associated Lagrangian.

Instead of interpreting u as a velocity, one eliminates it in favour of p before applying the first canonical Hamilton equation (v = dE/dp) which constitutes a dynamical definition of velocity. Thus, one immediately shows that in Cartesian dynamics, the velocity v is discontinuous at the origin (p = 0). Indeed, on eliminating u from the above couple of equations associated with Descartes dynamics one is led to $E = c\,|p|$ from which one deduces

$$v = dE/dp = c \text{ for } p>0 \text{ and } -c \text{ for } p<0. \qquad (51)$$

Because of the discontinuity, it is impossible to form a Lagrangian from which one usually deduces the two conservation laws needed in the resolution of practical dynamical problems.

Distinction between "physical admissibility" and "dynamical admissibility".

Contrary to Huygens dynamics which appears as physically admissible since it is compatible with Hamilton-Lagrange formalism, Descartes dynamics is not physically admissible. This dynamics constitutes a typical example where one has a "dynamical admissible" framework associated with a "physical inadmissible" one. The dynamical admissibility of the Cartesian framework will be proved by the adoption of a point of view that differs from those of Huygens and Lagrange-Hamilton. In brief, one may say that, as long as one interprets the parameter u in Descartes dynamics as a velocity (u = v = dE/dp, "usual rationality") or as a rapidity (u = w such that p = dE/dw, "emergent rationality"), one should admit the incoherence of this dynamics. However, because of the existence of other degrees of freedom provided by the Leibnizian methodology we shall be able to examine the situation without imposing a predetermined point of view but letting the Leibnizian principle determine the existence or the non existence of an appropriate point of view compatible with dynamical relativity.

(iii) Rehabilitation of Descartes dynamics by use of the Leibnizian formalism.

If one adopts the extended Leibnizian framework associated with a multiplicity of degrees of freedom each of which corresponding to one point of view, then one realizes that Descartes dynamics is compatible with the relativity principle. In this case, one should take into account the µ-derivative ($d_\mu/dv_\mu = D_\mu\, dE/dv_\mu$) also called extended derivative (developed in Appendix A and used all along the first part of this work). After having fixed one point of view that one identifies with u (say µ = f with v_f = u), then the couple associated with the Cartesian dynamics takes on the following form:

$$(E = cm|u|,\ p = D_f\, dE/dv_f = A(u)\, dE/du = mu) \qquad \text{(Leibniz-Descartes)} \qquad (52)$$

from which one deduces $A(u) = |u|/c$. Here the deviator $D_f(u) = A(u)$ plays the role of an absorber of discontinuities since it is responsible for the decomposition of the discontinuity:
$u/|u| = +1$ for $u > 0$ and -1 for $u < 0$ into two parts: $A(u) = |u|/c$ and $B(u) = u/c = p/mc$, so that the discontinuity is revealed only when the following relation is considered:

$$A(u)/B(u) = +1 \text{ for } u > 0 \text{ and } -1 \text{ for } u < 0. \qquad (53)$$

Having determined the adequate point of view through the determination of $A(u) = |u|/c$, one applies the process of derivation a second time as required by the Leibnizian principle of dynamical relativity obtaining thus the following remarkable result

$$I = \{A(u)\, d/du\, [A(u)\, dE/du]\} = m\,|u|\,/c = E/c^2 \qquad (54)$$

This last expression shows that the relativity principle is satisfied since the second extended derivative and all upper order extended derivatives do not lead to any new conservation law. In addition, let us note that contrary to Huygens parabolic dynamics which is formally equivalent to that of Newton, **Descartes dynamics does not only satisfy the dynamical relativity principle but it also reveals a proportionality relation between inertia I and energy E, characteristic feature of Einstein's dynamics ($I \equiv M = E/c^2$).**

Existence of an infinite number of possible dynamics out of which only those of Descartes and Huygens are dynamically admissible.

In order to place in evidence the strong constraint imposed by the Leibnizian dynamical relativity principle, let us note the rareness of the dynamically admissible solutions among the infinite number of the following possible dynamics: ($E = a_k m|u|^k$, $p = mu$) including those of Huygens and Descartes which correspond to k = 1 and 2 respectively. Indeed, a simple calculation that we do not reproduce here shows that only these two dynamical frameworks satisfy the dynamical relativity principle. All the others are dynamically inadmissible. This result is significant since it shows that in spite of the rareness of the functions (among an infinite number), that satisfy the relativity principle, the Cartesian one is among the few that verify this principle although this dynamics is incomplete and invalid for u = 0 (state of rest).

Comment on the historical form of Descartes dynamics.

Let us note that unlike Huygens conception of motion, Descartes one is less obvious. It is not clear to what extent did Descartes adhered to p = mu which is due to Huygens. Strictly speaking, and according to historians and philosophers of science, Descartes proposed only the following relation $Q = m|u|$ considered to be a conserved quantity associated with active substance that he called "quantity of motion". The active substance $Q = m|u|$ that Huygens replaced by mu^2 (vis viva or living force) is intimately related to the notion of energy so that in modern terms, Descartes quantity of motion is equivalent to $E = cm|u|$ (a conserved entity is defined up to a multiplicative constant factor). We shall show that even if the couple ($E = cm|u|$, $p = mu$) is a somehow modified version of the strict Cartesian dynamics where only $m|u|$ is due to Descartes, this couple imposes itself if one admits the principle of dynamical relativity to which Descartes adhered (at least qualitatively). To see this, let us replace the second expression $p = mu$ or equivalently $p/mu = 1$ by some arbitrary but finite form: $p/mu = f(u)$ before examining this new expression through the relativity requirement. In proceeding in this way, we get closer to Descartes dynamics since the finite function $f(u)$ is arbitrary and yet undetermined. Its determination is obtained by the application of the dynamical relativity principle (expressed in a differential form) that leads to $f(u) = [1 + K/u^2]^{1/2}$. In spite of the degrees of freedom provided by the integration constant K, only $K = 0$ turns out to be admissible. To see this, let us note that when $u \to 0 \pm \varepsilon = 0^{\pm}$ (since the dynamical relativity principle does not apply to $u = 0$ because of the irregularity of the Cartesian form $m|u|$), one gets $f(u) \to [1 + K/0^{\pm}]^{1/2}$ which may be **finite** only for $K = 0$. This clearly shows that even if one replaces 1 by $f(u)$ finite for any motion ($\forall$ u), getting thus closer to Descartes incomplete dynamics, the principle of dynamical relativity imposes $f(u) = 1$ so that $p = mu$ is not assumed anymore but deduced. In brief, one may say that if one assumes that $Q = m|u|$ corresponds to a conserved quantity at some scale, then one shows that the only other complementary conserved quantity should be $p = mu$. Any other proposition leads to a violation of the dynamical relativity principle.

III-3. Principle of simplicity versus the principle of relativity.

In this section, the attention will be drawn on the efficiency of the principle of simplicity whose nature is structural and mathematical while the principle of relativity is conceptual and physical. The principle of simplicity played such a major role in the development of physical science in general and mechanics in particular that it deserves a special attention. The latter will be examined and compared to the principle of relativity especially that both principles will coincide in so far as Newton and Einstein dynamics are concerned. On looking for a mathematical structure apt to conciliate the Newtonian framework with the Cartesian one, (both initially induced from experimental evidence), one discovers that the continuous and analytical framework associated with the principle of simplicity leads to a solution structurally equivalent to that of Einstein's dynamics. In addition, it turns out that only the Cartesian framework will impose a substantial constraint on the mathematical structure. Indeed, as already noted, any even function leads locally to a dynamics of the Newtonian type so that no constraint is imposed by the Newtonian world (except for the parity requirement which is also verified in the Cartesian dynamics). The Cartesian framework will impose a strong constraint intimately related to its irregularity. If one admits, with Leibniz, that the irregularity of Descartes dynamics reflects its local validity and that there exists a global regular form that reveals the local and irregular Cartesian framework, then one realizes that this may constitute a strong constraint. More precisely, there exists only a specific regular class of solutions that may include locally Descartes dynamics. These solutions may be expressed as follows:

$$Y = [\sum a_{2n} X^{2n}]^{1/2N} = [a_0 + a_2 X^2 + \dots + a_{2N} X^{2N}]^{1/2N} \qquad Y = E/mc^2 , \quad X = p/mc \qquad (55)$$

where N indicates the maximal value that n may take. It is readily checked that for great values associated with X, one gets $Y \to [a_{2N}]^{1/2N} |X| = A |X|$ so that one recovers Descartes irregular form. With or without such a constraint the Newtonian solution is recovered locally since any even function leads to a parabolic form in the vicinity of the origin. One easily verifies here, that for small values of X ($X \to 0$) the above equation leads to a parabolic (Newtonian) solution ($Y \to BX^2 + C$) where the constants B and C are expressed in terms of a_0, a_2 and N. It is remarkable to note that while Newton's dynamics does not impose any constraint Descartes dynamics imposes the above specific form, so that the irregularity usually considered as a vice turns to be a virtue. In particular, it should be noted that one may formally write a development in series with respect to the modulus or absolute value of X, but one should realize that such a formal structure is to be avoided since it adds other irregularities to the initial Cartesian one. This violently contrasts with Leibniz assertions about continuity and analyticity that may explain the discontinuous and the non-analytic forms as limit cases of some higher regular order. Thus, the inclusion of the constraint imposed by Descartes dynamics as a limit case leads to such a restriction that one may get the final regular solution without having to use the principle of relativity, but only the principle of simplicity. Indeed, if one considers the simplest solution that leads locally to Newton's dynamics on one side ($X \ll 1$) and to Descartes dynamics on the other side ($X \gg 1$), then one should select among the different values of N the one associated with N = 1 as shown later on through the application of the principle of simplicity stated in the next paragraph. Thus, one is led to a hyperbolic structure: $Y = [a_0 + a_2 X^2]^{1/2}$ specific to Einstein's dynamics. This shows that the combination of a few constraints borrowed from the realm of empiricism may lead to remarkable results. Here lies the secret of **efficiency without any intelligibility**. Many mathematical methods associated with some empirical evidence have shown their efficiency so that numerous modern physicists do not look for intelligibility anymore convinced that there is nothing to see behind the curtain. In the realm of dynamics and after having accounted for the principle of relativity for a short period of time and particularly with Huygens and Leibniz, most physicists followed the Newtonian belief in the existence of an absolute space in spite of the compatibility of his dynamical framework with Galilean relativity that was interpreted as an approximate principle and not a fundamental one. This conception, rooted in the physical community during the Newtonian period, was deeply reviewed in 20th century physics. The privilege of one conception rather than another is possible as shown above, since one may deal with the same dynamical structure, evoking for example simplicity instead of relativity. However, thanks to the works of Lorentz, Poincaré and most of all Einstein, the relativity principle (initially evoked by Galileo and others such as Descartes Huygens and Leibniz) was rehabilitated and it is now considered as the cornerstone of physical science, although it has never been developed in its most general form as shown in this work.

Statement of the principle of simplicity.

The "principle of simplicity" asserts that when dealing with a complicated situation and in the absence of physical constraints leading to a well-determined solution, then one should consider the simplest form among the different available possibilities. Obviously, the principle of simplicity is not absolute but relative to a framework full of undetermined possibilities. Here, this principle applies as follows: noting that the larger the exponent N, the more indeterminate the dynamics and the more complicated the expression, one is led to the less undetermined structure by selecting among the different undetermined coefficients a_{2n} ($n \neq 0$) only one. The only possibility compatible with both Huygens and Descartes dynamics corresponds to N = 1.

Remark associated with measurement.
Let us notice that one may single out N = 1 without any need to account for Huygens dynamics but only for the most regular solution for a fixed but arbitrary value of N. If one considers the above simplicity criterion keeps only one coefficient (among the numerous ones given above) leading to $Y = [a_0 + a_{2N} X^{2N}]^{1/2N}$ with N arbitrary, then one notices that this function becomes quasi-irregular in the vicinity of X = 1 (quasi-angular point) when the coefficient N takes great values N>>1. The increase is too slow when X < 1 while it becomes too large for X > 1. This poses a problem with measurement as shown in detail in the last paragraph of Appendix H. The most appropriate number to be considered corresponds then to N = 1.

It is remarkable to note that the passage from possible dynamics to the actual one may be obtained here either by use of the principle of simplicity or the principle of relativity. However, one should recognize that only the **principle of relativity** deserves to be considered as a **physical principle**, where **intelligibility** and **efficiency** go hand in hand, leading to both **explanation** and **exploration**. Since the principle of simplicity is purely structural and the elimination of the coefficients is performed without any sufficient reason, one renounces to any understanding of the physics lying behind the efficiency of mathematics. We have here a typical example where efficiency can be obtained independently of any intelligibility. According to Leibniz, the principle of simplicity may be of some use, at least in a first investigation, but it should be complemented by the principle of **sufficient reason** otherwise one explains "obscurius" by "obscurantum". Here the sufficient reason is provided by the relativity principle. Leibniz used to evoke these considerations metaphorically inviting us to close one eye, in a first step, in order to take a sight in a particular direction, without forgetting to widely open both eyes in a second step. If the first step may be sufficient for exploration, the second step is necessary for explanation.

III-4. Remarkable properties and efficiency of the Leibnizian methodology.

Remarkable properties:
We shall focus the attention here on two remarkable interconnected properties linked to the principles of dynamical relativity and continuity. To this end**, it should be emphasized that the Leibnizian methodology does not only constitute a weak formulation allowing to judge the dynamical admissibility of a given dynamics, but it also proposes a remedy to some anomalies** such as those associated with the absence of the analytical character or of continuity. This is a direct consequence of the differential form through which the Leibnizian methodology is expressed. By use of different constants of integration, one may get different possible dynamics (with or without discontinuities) associated with the same differential equation. This is precisely the case between Descartes and Einstein's dynamics were both of them are subject to the same differential equation, so that the difference lies only in the choice of the integration constants. This is also true for more general dynamical frameworks, such as the ones developed in the second part of this work, and related to some of the presently available dynamical approaches encountered in the framework of "doubly special relativity". In brief, one may say that the Leibnizian procedure plays the two different roles of a **judge** and of a **physician**. The first role consists in selecting the admissible solutions, and the second one allows, among other things, the replacement of **local discontinuous** solutions by **global continuous** ones. It also permits the replacement of local degenerate solutions by global non degenerate ones. If the attention was

mainly drawn on the problem of the passage from a discontinuous framework to a continuous one in the third part, one should recall that the same procedure (using integration constants) applies to the passage from a local continuous degenerate case to a global continuous non degenerate one. This is exemplified by the passage from Newtonian to Einsteinian dynamics. To see this, let us recall that the parabolic Newtonian structure can be said to be degenerate, in the sense that any even and regular function leads locally to a parabolic form (Newton: $E = p^2/2m + E_0$). However, if one expresses this parabolic form as a differential equation, then one may write the following relation: $p'' + m^2/p^3 = 0$, obtained from Newtonian dynamics as follows:

$$p' = dp/dE = m/p\ , \quad p'' = d^2p/dE^2 = d(m/p)/dE = [d(m/p)/dp][dp/dE] = -m^2/p^3$$

Because of the degrees of freedom provided by the integration constants, this second order differential equation leads either to the initial parabolic solution or to a hyperbolic one (Einstein's dynamics) or still to other possibilities (doubly or deformed special relativity) according to the imposed limit conditions and the interpretation of the constant m.

Efficiency of the Leibnizian methodology.
At first sight it appears mysterious that without any knowledge of modern physics, Leibniz was able to mention the existence of a framework capable to enclose Huygens and Descartes dynamics in a higher unity. However, if one looks at the different centres of interest of Leibniz among which the catenary's curve (see Appendix E) and the study of conics, then one discovers that the mathematical structures are exactly the same as those dealt with through the passage from 17^{th} century to 20^{th} century dynamics. In particular, one easily shows that in the following hyperbolic structure $Y = \{[A^2 + X^2]^{1/2} - A\}$, Y is proportional to X^2 when $X^2 << A^2$ and to $|X|$ at the other extreme ($X^2 >> A^2$) so that one recovers Huygens and Descartes structures, respectively, as two approximate and local expressions of a unique global solution. The **three different solutions** may be cast into a **unique differential equation** so that each solution corresponds to a specific choice of the constants of integration. Such methods were much emphasized by Leibniz as shown extensively by different authors and particularly by the work of M. Parmentier [9], who examined numerous mechanical and mathematical situations developed by Leibniz, on technical as well as on epistemological grounds. This presentation played a major role in the development of the present Leibnizian methodology.

III-5. Motivations in favour of the rational and relational Leibnizian methodology.
It seems to me important to trace some lines of thought that led me to the rehabilitation of the Leibnizian paradigm. Apart from the above mentioned work provided by Parmentier [9], the article written by C. Comte [10] concerning a direct link between Leibniz and relativity, and another indirect one corresponding to a synthesis discussed by Lévy-Leblond [13] who focuses the attention on different points of view on motion, there are at least four other complementary reasons that led me to work in the rational and relational Leibnizian framework that differs substantially from the usual one inherited from Newton and rationalized by Lagrange, Hamilton and their followers. These four reasons given below are decomposed as follows: *(i) subjective qualitative reason, (ii) subjective quantitative reason (iii) objective qualitative reason and (iv) objective quantitative reason.*

(i) subjective qualitative reason*:* The subjective qualitative reason is intimately linked to an assertion due to Kurt Gödel, according to which the Leibnizian paradigm is not dead as usually believed, but its development requires a more subtle framework, semantically superior to the one adopted by the physical community who followed the Kantian distinctions between physical and metaphysical approaches. If this reasoning is not only subjective but also qualitative, it is because Gödel did not work directly on dynamics but on the scientific method as a whole. This line of thought was also developed at different degrees by the mathematicians René Thom and Alfred North Whitehead. The few discussions I had with René Thom in Paris and Besançon after the lectures he delivered on biology, physics and epistemology with their associated methodologies (among which the Lagrange-Hamilton formalism) constituted a germ to my investigations for a better understanding of the Lagrange-Hamilton formalism and its possible inclusion in a wider framework. Even if René Thom did not investigate the specific problem of dynamics as done in this work, his qualitative methodology and his private discussion with professor Avez (one of my professors: "course on differential forms") on the relevance and necessity of the Lagrange-Hamilton formalism played a non-negligible role in my interest to this subject.

(ii) subjective quantitative reason*:* Unlike the above mathematicians, Hans Reichenbach has developed a Leibnizian interpretation of Einstein's dynamics, asserting that Leibniz was a forerunner of Einstein's ideas (Ref.[4] page 42) . In his book on space and time [17] Reichenbach writes : "*It is the more remarkable that Leibniz, this genuine philosopher, was able to understand the nature of scientific knowledge to such an extent that, two hundred years later, a new development of physics and an analysis of its philosophical foundations confirmed his views*". This assertion encouraged me to look for a deeper understanding of Leibniz ideas on the possible existence of a higher dynamical rationality than the one provided by the conventional analytical models. Being convinced that the study of Leibniz may lead to a more profound analysis of motion contrary to what is usually believed, and having spent many years working on d'Alembert's principle (applied to complex media), whose origin is directly linked to Leibniz "vis viva", I went looking for objective arguments in favour of Leibniz's methodology. Two main reasons attracted my attention as shown below.

(iii) an objective qualitative reason*:* There is an objective qualitative reason that shows the inconsistency and the logical deficiency of the usual analysis of Leibniz's approach of possible worlds when it is compared to the world of physical science. Many researchers at the frontiers of mathematical physics and philosophy make illegitimate links, the most famous of which being the association between Leibniz's "best of all possible worlds" and the "least of all possible actions" at the basis of Lagrange-Hamilton formalism. As shown in the preamble, such an association is not acceptable. In the first case, the selection of the "best" operates on families of curves (tree like structures) where the different curves (branches) constitute the points of view as shown in (18), while in the second case the selection of the "least" operates on simple curves (branches), so that one confuses a "whole" (tree) with a "part" (branch) which is a logical inconsistency.

(iv) an objective quantitative reason*:* The first ten years of my research activity was centred on the positive work of d'Alembert through his "principle of virtual power", rehabilitated in the French mechanical school particularly by professors Germain and Maugin, and applied to complex media (electro-magneto-thermo-mechanical interactions), including irreversible and dissipative processes as well as singular surfaces and interfacial properties. The above mentioned Gödel and Reichenbach opinions on Leibniz's methodology helped me to be more critical with regard to d'Alembert's negative opinion on Leibniz's metaphysics, opinion shared by most physicists. Thus, after having followed the line of thought developed by my teachers as to the importance of d'Alembert's principle as compared to Newtonian an Lagrangian antagonistic methodologies, I became interested in the **negative part of d'Alembert's work**, and in what he could not achieve because of some logical errors that he made in the evaluation of the famous "vis viva controversy", and particularly concerning the rejection of Descartes non analytical dynamics. Some of these errors have been perpetuated since, because physicists as well as philosophers and historians of science dealing with dynamics reproduce the same contradiction, (developed in Appendix D). It deals with the fact that Descartes dynamics has been rejected while Newton's one was retained. From a logical point of view, and after the development of the generalized Leibnizian framework, one is led to the following conclusion: either one rejects both dynamics replacing them by another more powerful one, if one considers that only the global approach counts; or one accepts both of them as two local frameworks, each of which constituting one facet of a coin. More precisely, one is valid in the vicinity of the rest state and the other far away from this rest state. When I looked at the problem from a conceptual stand point, it was a big surprise to me to discover that, not only Descartes dynamics (interpreted in the light of Leibniz's methodology) is compatible with the generalized dynamical relativity principle, but it also leads to the famous mass-energy equivalence specific to Einstein's dynamics as shown earlier. This fact was never placed in evidence, not only because Descartes dynamics is an old story forgotten by most physicists but also and above all, because such a framework requires weakening the physical hypotheses in two directions. Firstly, one should be able to go beyond the Lagrange-Hamilton formalism (which is at the basis of the rationality of space-time physics as asserted by Noether's theorem), then one should be able to deal with the relativity principle in a wider framework than that of the analytical continuum since Descartes dynamics is non analytical. Thus, one has to surmount **two epistemological obstacles**, one associated with the concept of **motion in space-time**, and the other associated with the **continuity hypothesis**.

III-6. Qualitative versus quantitative.

Physicists usually consider that Leibniz's approach of natural phenomena is the kind of method that one should not adopt in physical science because of its qualitative character and lack of precision. This method, based on the idea that "quantitative is poor qualitative", violently contrasts with the usual method developed by physicists and inherited from Newton, d'Alembert, Lagrange and their followers. This is exemplified in Ref. [24], in the chapters dealing with space and time, where the author asserts that with the qualitative Leibnizian method one cannot reach any positive and predictive science even if at first sight and from a logical point of view this qualitative methodology seems irreproachable. More precisely the author writes: "*Time is the abstract of all relations of sequence, wrote Leibniz, so complementing his other statement … space is the abstract of all relations of co-existence. These are neat forms of words, and they have the ring of truth in them, but they do not advance the understanding of the physicist in any positive way*". The Leibnizian approach considers "quantity" as inferior to "quality" in the sense that the confrontation between qualitative principles is capable to produce quantitative results. Such an assessment is not encountered in dynamical approaches since the basic postulates are anchored in a quantitative framework. This is not only present in empirical approaches or in the

Lagrange-Hamilton formalism whose aim was to provide a rational framework to space-time physics, but appears also in modern theories associated with the "emergent rationality" such as the one developed by C. Comte based on weak assumptions accompanied by an optimization of the number of postulates. Indeed, C. Comte writes in his Thesis: **"Doing physics, is (first of all) elaborating progressively precise definitions of the used terms, in inventing experiences where they intervene in relation to measurable quantities".** (My translation). Following this line of thought the author explains the interest of dealing with an additive parameter called rapidity justified from both experimental and theoretical stand points. This anchorage in measurement eliminates from the start any possibility of dealing with an a priori infinite number of points of view on motion and consequently of any qualitative definition of motion. In stating precisely what motion should be, one is directly led to the realm of quantity eliminating thus the possibility of operating on a number of qualitative undetermined entities. Such an elimination leads to the evacuation of a number of considerations such as those associated with inter and/or trans-subjectivity. To deal with such correlations, one needs at least recognize their possible existence inside the same theoretical framework. This is neither the case in the "usual rationality" nor in the "emergent" one.

III-7. Some fruitful, intuitive, structural and analogical ideas associated with broken parity, finiteness of energy and a new conception of motion.

Let us underline some facts in direct relation with scientific discovery. In particular, let us note that if this work follows the line of thought developed by rational thinking, presenting things in an analytically articulated way, so that one may single out the basic postulates, the solutions were initially obtained by use of various structural, heuristic and analogical procedures, whose rational justification was not always obvious. Thus, as usually expected, **exploration preceded explanation** so that the different general solutions that extend the Einsteinian approach and the Lorentzian metric, which will be summarized below, were obtained long before I understood their articulation with the basic postulate on which the present approach is founded: "The principle of dynamical relativity". In the forthcoming development, the heuristic and analogical procedures will be recalled, and the path to the obtained solution will be traced qualitatively, showing how one can be led to the same physical solutions borrowing different avenues, some of which may be far from usual rationality.

Apart from the fact that Einstein and Newton dynamics may be cast into a unique differential form whose integration leads to both dynamics as well-as to more general ones (doubly or deformed special relativity), as shown above (first application of the second part of this work), three other correlated intuitive ideas played a major role in the development of the present Leibnizian context. One of these evoked earlier, concerns the structural intuition of a point at the basis of "inter-subjectivity" and "trans-subjectivity" (here, the state of rest) regarded not as a simple point but as an **accumulation point**: an infinite number of curves (each of which constituting one point of view on motion) governed by a recurrent series, coincide and converge towards such a point following a unique common tangent. This intuition played a major role in the construction of a formulation of a Leibnizian type (See Appendix H for more details as to the construction of an accumulation point).

The two remaining ideas concern 17^{th} century mechanics and the oscillator problem. More precisely, the first of these is associated with Newtonian dynamics, in its relation to conservation laws, while the second is associated with the analogies between mechanics and electric circuits through the “linear oscillator” model. These two ideas to which two paragraphs are devoted below, go back to the period I was still a student, but they were ill articulated since the reasoning was purely structural and/or analogical, lacking a physical justification. If the first contact with rational physics suggests numerous questions, most of the time the unanswered questions are due to a simple lack of knowledge. However, some of the questions may be profound and still unsolved ones for, the recognition of a pertinent question is not always easy to grasp. There is a zone of uncertainty: a question may be considered to be resolved by the scientific community while it is not really well understood. This will be illustrated by the well-known usual relation between Newton’s dynamics and Einstein’s one. The problem is usually considered to be totally resolved, and nothing new could be said on such an old subject matter. However, the present work shows that the understanding of this subject is fully renewed with the consideration of an infinite number of points of view on motion, obtained through a recurrent series that allows the passage from one point of view to the next by the simple application of a formula unknown before. This fact leads automatically to a major conceptual difference, as shown in this work through the introduction of “inter-subjectivity” and “trans-subjectivity”. In particular, the state of motion turns out to be associated with an infinite number of curves that coincide locally (weak motion) according to a unique tangent composing the trunk of a treelike structure and reflecting thus the local character of Newtonian dynamics. This fact differs substantially from the interpretations provided by the usual as well as the emergent rationalities. Before reaching a rational formalism concerning this idea of multiplicity of points of view, borrowed from Leibniz’s conceptual framework, I was faced with the following more immediate problem directly linked with Newtonian dynamics and its link to conservation laws.

(i) Newtonian dynamics and conservation laws.

When dealing with the link between Newtonian dynamics and conservation laws of elastic and inelastic collisions, one is struck by the harmony associated with these equations: $E = \frac{1}{2} mv^2 + U$, $p = mv$ and $M = m$. One immediately notices that p derives from E and M derives from p. ($p = dE/dv$, $M = dp/dv = d^2E/dv^2$) so that **the derivative does not play here a descriptive kinematical role as for the definition of the velocity ($v = dx/dt$) but it plays a dynamical one since it corresponds to a “generator of conservation laws”.** This leads firstly to a substantial economy of thought and secondly to a rational articulation since the knowledge of energy is sufficient to deduce the two other equations by a simple derivation with respect to the velocity. Last but not least, if one continues to derive indefinitely, one gets no more conserved entities but indiscernible null results. However, this harmony turns out to be only apparent and purely accidental because of the local character of Newtonian dynamics. Indeed, if one passes to Einstein’s dynamics, one discovers that the relations become more complicated and the harmony through the derivative is lost. Let us also note that one could also obtain the same conclusion by referring to the Lagrange-Hamilton formalism (at the basis of physical rationality) which shows that impulse does not derive from energy but from another entity called the Lagrangian. After having adhered to the usual rationality for some time in a rather absolute manner, I discovered later on that the first intuition was not insane but it requires a modification and an enlargement of what is meant by motion. This led me to the development of the part of this work associated with the multiplicity of points of view on motion where the notion of a derivative plays an essential role and its generalization allowed me to include the usual Lagrangian rationality in the present extended framework. The adoption of another point of view on motion was not initially related to my readings of the epistemological works of J.M. Lévy-Leblond and C. Comte (that I discovered later on), but to a certain rational procedure, where the derivative plays an essential role in the study of hyperbolic and trigonometric functions. This other use of the derivative was initially associated with the "linear oscillator" as shown below. My interest in the above epistemological works was motivated by something I had integrated before without a full understanding of their importance to the foundation of dynamics.

(ii) Dynamics in relation to the "linear oscillator".

It is well-known that the "linear oscillator" is linked to trigonometric functions, and represented by the following second order differential equation: $y'' + y = 0$ whose integration leads to $y^2 + y'^2 =$ constant. The simple replacement of the "plus" sign by a "minus" one, leads to hyperbolic functions governed by $y'' - y = 0$ or by $y^2 - y'^2 =$ constant, after integration. This hyperbolic form may be directly linked to Einstein's dynamics $E^2 - p^2 = E_0^2$, provided one admits that the impulse p derives from energy with respect to some parameter not to be confused with the velocity v but with the rapidity w as shown in this work ($p = dE/dw = E'$). This corresponds to $E = E_0 \cosh w$, $p = E_0 \sinh w$ (c = 1: natural system of units). This analogy suggests the **existence of another possible rationality provided one admits that motion can be accounted for through a different point of view**. This other perspective turns out to be structurally similar to the one provided by what we have called "emergent rationality". The authors who deal directly with dynamics start their investigations by replacing the non additive composition law of velocities by an additive one on which all the rest is based leading thus to the rapidity parameter. Here, the attention is focused on the structure of trigonometric and hyperbolic functions when these are looked at from a differential point of view. These are then distinguished by a simple difference in sign as mentioned above. The emphasis here is not only put on the fact that the analogy with the harmonic oscillator turns out to be fruitful since it shows that one may be led to another form of rationality, but **the link to the linear oscillator suggests possible extensions that deserve to be transferred to fundamental physics**. One may examine the hyperbolic counterpart of the damped linear oscillator, where in addition to the second order derivative, a first order one enters into play to account for damping or dissipation. Obviously, in the absence of the coefficient associated with the first order derivative, one recovers the initial solution where no damping is considered. On applying the same well-known method to dynamics, one obtains solutions of the following forms:

$$E = E_0 \exp(Sw) \cosh(w) \qquad p = E_0 \exp(Sw) \sinh(w) \tag{56}$$

or equivalently,

$$E = [(1+v)/(1-v)]^{S/2} \{E_0/[1-v^2]^{1/2}\} \qquad p = [(1+v)/(1-v)]^{S/2} \{E_0 v/[1-v^2]^{1/2}\} \tag{57}$$

provided that one relates rapidity w to velocity v through $v = \tanh w$. Noting that we have $v = p/E$, Eqs.(57) turn out to be associated with the following dynamical and kinematical fundamental relations:

$$(E - p)^{1+S} (E + p)^{1-S} = E_0^2 \qquad (dt - dx)^{1+S} (dt + dx)^{1-S} = d\tau^2 \tag{58, a}$$

where we have set

$$p = E_0 dx/d\tau \qquad E = E_0 dt/d\tau \tag{58, b}$$

In Appendix P, the question is dealt with more explicitly and in connection with a recent work [33] where the problem is approached through the "usual rationality". Obviously, in the absence of the scale term (S = 0), responsible for parity breaking, one recovers Einstein's dynamics with its Lorentzian metric.

<u>*(iii) Simultaneous account for broken parity and finiteness.*</u>
Let us note that the account for finiteness of energy in addition to parity breaking, leads to

$$(E_{kl} - p_{kl})^{1+S} (E_{kl} + p_{kl})^{1-S} = E_{0kl}^2 \qquad (dt_{kl} - dx_{kl})^{1+S} (dt_{kl} + dx_{kl})^{1-S} = d\tau_{kl}^2 \tag{59}$$

This is easily obtained if one recalls the following dynamical relations

$$E_{0kl} = E_0 / [\,1 - (E_0/E_M)^k]^{l/2} \qquad E_{kl} = E / [1 - (E/E_M)^k]^{l/2} \qquad p_{kl} = p / [1 - (E/E_M)^k]^{l/2} \tag{60, a}$$

with

$$(k, l) = \{(1, 1)\,;\,(1, 2)\,;\,(2, 1)\} \tag{60, b}$$

[Appendix L through (L12), (L13)and $(L16)_{2,3}$]. Their kinematical counterparts correspond to

$$d\tau_{kl} = d\tau / [\,1 - R^k]^{l/2} \qquad dt_{kl} = dt / [1 - (Rdt/d\tau)^k]^{l/2} \qquad dx_{kl} = dx / [1 - (Rdt/d\tau)^k]^{l/2} \tag{61}$$

These are obtained by use of Eqs.(58, b) as well as wave-particle duality [see Eq.(32)]

$$R = E_0/E_M = k_0\, x_m = \omega_0\, t_m = E_0\, t_m \tag{62}$$

which links the energy concepts ($E_0<E<E_M$) associated with minimal and maximal energies E_0 and E_M, to that of frequency ω_0 or wave number k_0, and space x_m or time t_m by use of wave-particle duality as shown explicitly in the first application of the second part of this work (Planck's constant and light velocity are identified with unity). Let us recall that the passage from infinite to finite energy considerations have been dealt with in conformity with the dynamical relativity principle. Let us also emphasize the fact that when dynamics precedes kinematics, one is able to determine directly the class of kinematical solutions. If one postulates kinematics as usual in conventional approaches one is then limited to too simple structures since one has to tackle with the structure of space-time and that of relativity at the same time. **Whenever one deviates from Galilean or Lorentzian metrics it becomes tedious to find out the appropriate structure especially that, as shown above, the new coupled metrical structures do not enter in the realm of Riemannian quadratic geometry, (see Appendix P for more details).** This fact led some scientists to propose extensions of Lorentzian metrics which are more or less arbitrary in the sense that they are not based on the principle of relativity.

(iv) Analogical thinking related to empirical and rational approaches.

The analogical way by which the possibility of this rationality has been discovered and which turned out to be structurally equivalent to the "emergent rationality", differs substantially from the one usually adopted in physics, operating directly on dynamics. This analogical way in dealing with physics has shown to be fruitful at different occasions and at the most fundamental level. In this regard, let us recall that the analogy between mechanics and electromagnetism is historically at the basis of the discovery of Einstein's dynamics. In the same manner as the Lorentzian metric was transferred from electromagnetism to dynamics, here the more general structure which, strictly speaking, is not metrical anymore, is obtained by analogy with the linear damped oscillator. Obviously, after a discovery, obtained through analogy, one should interpret things differently in order to avoid the semantic associated with the initial structure, which plays the role of scaffolding and should be removed after the construction has been achieved. [One famous historical example is the one attached to the so-called "velocity of light" (historical accident) induced from electromagnetism and attached to Einstein's approach].

The different methods adopted by use of analogy and/or philosophy, explain why the obtained results differ from those adopted by the usual four kinds of different researches in physics: empirical, theoretical, mathematical and epistemological. The **empirical physicist** realizes that the notion of velocity is **not operational** anymore in high energy physics because of its finiteness and of its asymptotical behaviour (no measurement of difference between velocities is possible for very high energies). He is then obliged to change of point of view in so far as the measurement of motion is concerned. This is one reason for which the rapidity concept was introduced.

The **theoretical physicist** shows that the replacement of the **non additive** composition law associated with the velocity ($v' \neq v + V$) by an **additive** parameter ($w' = w + W$) leads also to rapidity. The **mathematician** realizes that the **natural** parameterization of the hyperbolic structure ($E^2 - p^2 = m^2$), a characteristic feature of Einstein's dynamics, corresponds to $E = m \cosh w$, $p = m \sinh w$. Again one discovers rapidity.

The **epistemologist** dealing with physics in relation to geometry, philosophy and history of science knows that the passage from the Euclidean concept of space to the hyperbolic one leads to a certain structural deformation of Euclidean space. In particular, the hyperbolic counterpart of the following Euclidean formula $2\pi r$ corresponds to $2\pi R \sinh(r/R)$. On setting $w = r/R$ one gets again an analogy with the notion of rapidity. [The parabolic form of Pythagoras theorem: $a^2 + b^2 = c^2$ transforms into the hyperbolic one $\cosh(a/R) \cosh(b/R) = \cosh(c/R)$].

All these different adopted approaches converge towards the same entity (rapidity replacing the velocity). If one adopts Leibniz's association of the multiplicity of points of view to "monads" one may say that what Leibniz calls **"dominant monad"** corresponds to **rapidity,** since this concept constitutes the **central point of view** with regard to the hyperbolic structure at the basis of Einstein's dynamics. As pointed out by J.M. Lévy-Leblond and C. Comte, an interesting way to deal with motion quantitatively and directly (without any consideration of unnecessary entities such as the Lagrangian), consists in using the rapidity parameter in connection with a methodology embedded in group theory. Different models dealing with such a connection are available [14-16].

One should however keep in mind that in spite of its central character, the rapidity (dominant monad) is only a point of view while **Leibniz's ideal is to get beyond any point of view.** He was looking for a higher principle apt to generate different points of view, each of which associated with a specific remarkable property.

Although the present work is somehow anchored in reflections intimately related to the epistemological works of J.M. Lévy-Leblond and C. Comte, let us recall that epistemology is above all concerned with the foundations of a theory on a firm ground, avoiding all superficial constraints that reduce the bearing and the full significance of the approach. This explains why the pedagogical and foundational works of J.M. Lévy-Leblond and C. Comte deal neither with the question of broken parity nor with that of finite energy. Thus, in addition to the difference in the adopted methodology, there is also a difference in the pursued goal. **Our goal is not only to present a better approach to dynamics avoiding any dogmatic, unnecessary hypothesis but also to propose possible extensions for the future (**without forgetting missed past opportunities, through a better articulation and a more rational link with pre-Newtonian dynamics).

III-8. Some ideas linked to Leibniz's methodology (Comte, Lévy-Leblond, Kant).

Comte's methodology: economy of thought and structural simplicity.

C. Comte considered the problem of dynamics in its generality proposing not only pedagogical contributions but also and most of all an economical line of thought (avoiding any non-necessary entity such as the Lagrangian, at the basis of rational physical science). According to C. Comte (who adopts Langevin's conceptual framework and partially some Leibnizian ideas among which the priority of dynamics on kinematics) it is not only possible to approach relativity from a dynamical standpoint before deducing kinematics at a later step but, it is also preferable not to presuppose any particular scheme of organisation of nature as the one given through the least action principle (Lagrange-Hamilton formalism). These ideas (drawn from Ref.[16]) are justified by the development of an autonomous dynamical approach of relativity based on group theory, associated with the rapidity parameter. This parameter is, according to C. Comte, "the best one" through which one deals with dynamics. Two things are meant here by the "best": economy of thought and structural simplicity. Both, **economy of thought** (no need to any Lagrangian denounced by Comte as a [non-necessary] particular scheme of organization of nature) and **structural simplicity** (the rapidity composition law is the simplest one for it is additive unlike the velocity or any other point of view) are satisfied. C. Comte favours thus a new rationality called here "emergent rationality", following the line of thought initiated by Langevin. His final aim is to apply this same rationality to quantum mechanics hoping to erase the rupture between classical and quantum physics (as indicated by Comte at the end of Ref.[16]).

Lévy-Leblond's methodology and its link to the multiplicity of points of view.

J.M. Lévy-Leblond is conscious of the interest and importance of rapidity (associated with group theory), but he remains open to the multiplicity of points of view on motion. This can be seen through the examination of his articles on this subject. In particular, he focuses the attention on the independence and importance of special relativity with respect to electromagnetism when he writes (Ref.[11]): *"the logical ordering of the theoretical foundations here is inverse to the chronological order of discovery; "relativity" nowadays is to be thought of as a general theory of space-time, which acts as a constraint, a "super-law", on all specific physical phenomena taking place in this common arena. Electromagnetism thus has to be built upon special relativity rather than the reverse"*. Moreover, after having championed the idea of multiplicity of points of view (presenting two different ones) he ends his paper with this sentence: *"I do not doubt that other point of views yet exist which might be worth developing"*. Four years later, he proposes a synthesis [13] in which he presents three different points of view on motion that he calls speeds distinguishing them by the denominations: velocity, celerity and rapidity. The introduction of Ref.[13] ends with the following sentence: *"By considering all three of them, most conceptual difficulties either vanish or are greatly reduced"*. This clearly shows that, for him, there is not an **absolute** "best way" to deal with motion, but only a **relative** one expressed at the end of the same article through the following sentence: *"rapidity is the most natural "speed parameter" to study changes in reference frames, i.e., relativity theory"*. In the introduction as well as in the main text, Lévy-Leblond puts the emphasis on other possible points of view on motion in the Einsteinian framework. In the main text, one reads "*The main conclusion to be drawn is that, within Einsteinian relativistic kinematics, there is no such thing as a concept of speed.* ***At least*** *three different quantities can be defined, which generalize the single Galilean quantity, each in its own way. It is but a proof of intellectual inertia that a consideration of this multiplicity has been too long prevented. To endow one of these quantities with the usual name "velocity" used for the Galilean quantity, is an abuse of language, probably unavoidable, but which should nonetheless be recognized"*.

To some extent, one may say that Lévy-Leblond is rehabilitating a line of thought developed by Leibniz long ago, but forgotten since the advent of Newtonian dynamics with its unique concept of "velocity" associated with motion. However, unlike Leibniz's belief in a higher rationality capable to encompass the above-mentioned three points of view adding to them other ones, Lévy-Leblond remains directly connected to the Kantian paradigm extended beyond the structure of Newtonian mechanics on which it was initially built. According to this paradigm in its extended version, one needs a non-conceptual element such as the notion of velocity, celerity or rapidity to account for dynamics. Each parameter (leading to one point of view on motion associated with one dynamical approach) requires the selection of one remarkable property associated with a "simplicity criterion" on which dynamics is erected. Lévy-Leblond explains that the velocity v is the **simple ratio** of distance to time, (or dynamically speaking, a simple ratio between the two conserved quantities: impulse over mass-energy $v = p/M = p/E$, in natural units where $c = 1$) the celerity u (using the notation of the present approach) is associated with a **proportionality relation** with respect to impulse ($p = mu$) and finally the rapidity w is associated with an **additive composition law** ($w' = w + W$). Lévy-Leblond enumerates these three remarkable properties (simple ratio, proportionality relation and additive composition law) at the end of his concise but highly significant paper [13].

Link of the different methodologies with the Kantian paradigm(from the "how" to the "why").

The articulation of each remarkable mathematical property with one point of view on motion shows the intrusion of **mathematics** into physics at the most fundamental level. In this regard, let us note that the advent of the Kantian paradigm through the introduction of the "synthetic a priori" judgement was also based on some mathematical examples (probably borrowed from Kant's discussions with the great mathematician Euler who convinced Kant to follow Newton rather than Leibniz). As pointed out by Deleuze, in his course on Kant and Leibniz, a typical example of a "synthetic a priori" judgement is the one associated with the "straight line conceived as the least path between two points". For Kant, the concept of a straight line requires a mechanism of comparison which is said to be non-conceptual (*an a priori or pure form*) selected through the remarkable minimal property (least path). In spite of many significant differences, [Newton (linearity of impulse with respect to velocity or force with respect to acceleration), Maupertuis, Lagrange and Hamilton (least action principle), Einstein, Minkowski, Taylor and Wheeler (proportionality relation between celerity and impulse), Comte (additive composition law) and Lévy-Leblond's synthesis revealing these different remarkable properties], all these differences are internal to the Kantian paradigm inviting the physicists to build on a specific property considered as a non-concept, an a priori or a pure form. Such a remarkable property has to be imported from the outside and should not belong to the conceptual apparatus. Having admitted this major point associated with the relevance of "remarkable properties" each one leading to a model, in the construction of a physical picture of the world, one clearly understands why Leibniz cannot be considered as a physicist in the usual scientific acceptation of the word. Unlike Kant (*who replaced the Aristotelian pre-scientific distinction between "**essence**" and "**modalities of existence**" by the scientific distinction between "**concepts**" and "**pure or a priori forms"***), Leibniz remained attached to the previous Aristotelian distinction, but with a major different interpretation. For Leibniz, opposed to the majority of scholars, the modalities of existence are not to be conceived as simple appearances or pure illusions but as "well-founded phenomena" constituting a deformation or a simple **projection** of the underlying objective reality (essence). Kant was right in criticizing the pre-scientific distinction when it associates existence with a pure illusion that should be rejected in favour of essence. In inviting us to look at the **conditions of apparition, of a phenomenon** instead of looking at what is **hidden behind such an apparition** Kant brings an interesting solution, widely adopted by the scientific community. However, Kant's proposition does not prove the inexistence of the possibility of a higher intelligibility as claimed by Leibniz. It is simply because science concentrated its efforts on **"how"** do things appear in a certain way, rather than on **"why"** do these things have such a property". In other words, the remarkable property (on which one constructs a specific methodology) prevents one to ask the question of its "raison d'être". The Leibnizian principle of sufficient reason aims at looking for the reason hidden behind the existence of the different remarkable properties. For Kant as well as for the scientists who followed his methodology, (consciously or unconsciously), there is nothing hidden behind the postulated properties. The above-mentioned simple ratio, proportionality relation as additive character constitute remarkable properties imported from the outside, each one being associated with a specific methodology. These are not deducible from any hidden unifying principle as advocated by Leibniz. If Leibniz's epistemology is mathematically constructible and physically significant then, the existence of such a unifying principle reaches the "why" as well as the "how" while Kant's epistemology aims only at the "how" things appear under certain conditions (justified here by remarkable properties).

III-9. Specificity of the Leibnizian methodology (Science and culture).

Beyond simplicity and complexity.

It should be emphasized that an epistemology of the Leibnizian type, anchored in the distinction between "essence" and "modalities of existence", is far more demanding that the Kantian epistemology based on an a priori selection of "one modality of existence" at a time. The Kantian paradigm rejects the simultaneous existence of a multiplicity of modalities of existence governed by a unique principle. If one modality of existence may be accounted for through different principles, **one principle** (chosen according to Kantian paradigm) **cannot encompass different modalities of existence**. With the advent of what we have called "emergent rationality" different principles have been used to determine this modality of existence. The same holds for the two other points of view or modalities of existence (velocity and celerity). Different principles are available to grasp each one of them. The reciprocal is not true: the Kantian paradigm, adopted by the scientific community, does not lead to a unique dynamical principle apt to encompass different modalities of existence simultaneously. Once a principle is selected, it operates analytically with some parameter associated with motion. There is a **direct connexion** between a **specific rational methodology** and the **remarkable property** on which it operates. The only way out of this situation is to look for some methodology unconnected with a remarkable property whatsoever. Since any remarkable property is a reflection of the simplicity criterion associated with one modality of existence, the Leibnizian methodology should necessarily lie **beyond the question of simplicity and complexity**. This means that one should leave the realm of quantity to that of quality in so far as modalities of existence are concerned, since any remarkable property is embedded into a quantitative framework by its very nature. How then is it possible to make prediction if one is cast in the realm of quality? One faces here another problem, dealt with in another framework associated with the works of R. Thom. This mathematician (whose methodology was rooted in a qualitative framework of an Aristotelian type) showed that what one gains in intelligibility is lost in predictability. Here, the situation is different, in reason of the distinction between the two orders of "essence" and "modalities of existence". The answer to this question lies in the fact that only the modalities of existence are cast into the realm of quality. One may continue to operate quantitatively on essence without selecting any quantitative modality of existence. To do this one should necessarily distinguish between "essence" and "modalities of existence" or between an objective "one" and a subjective "many". Only then, one may attach qualitative considerations on the "many" and a quantitative principle on the "one". This crucial distinction is the one that was rejected by Kant assumed to be purely metaphysical (in a pejorative sense) and chimerical with no relevance to positive science and that we rehabilitate in the present work.

Leibniz's rejection: risk of extinction of science as a culture.

For a long time in the history of humanity, it was believed that women's brain was unapt to receive a scientific education; so that science was not taught to them. The examination of the history of science without awareness of the above sociological fact leads to the following discovery: very few women have contributed to physics initially called natural philosophy. One may abusively deduce that women are not sufficiently intelligent in this field of knowledge. The rare contributions produced by women are then considered as accidental and with no particular interest. It is this very scheme that applies to Leibnizian physics as compared to that of Newton. One starts by asserting that the Leibnizian conceptual scheme dealing with possible worlds and different points of view on each of them is not apt to deal with physics, then one discovers that on looking behind us, very few Leibnizian concepts are compatible with physical science. To examine the relevance of the Leibnizian conceptual apparatus, one should test it, instead of rejecting it abusively (outside of physics) from the start. With the exponential increase of the methods, recipes and tricks of the trade, it is urgent to look for some higher rationality and articulation between bits and pieces, as advocated by Leibniz. Otherwise, the scientific enterprise runs the **risk of extinction as a culture,** like those giant pre-historic animals, victims of their gigantism. To avoid such a danger, it becomes necessary to enlarge the following already mentioned assertion adopted by most physicists: **"Doing physics, is (first of all) elaborating progressively precise definitions of the used terms, in inventing experiences where they intervene in relation to measurable quantities".** With such a precise and quantitative definition, Leibniz's methodology (associated with trans-subjectivity and inter-subjectivity) cannot belong to physical rationality by its very nature. Firstly, it combines the quantitative and qualitative associated with the couple (essence, existence), [absent from the usual methodologies], as follows: (qualitative-qualitative), (quantitative-qualitative), (qualitative-quantitative) and (quantitative-quantitative) articulating thus in a subtle manner the orders of quality and quantity as shown in this work. Secondly, even in the last purely quantitative case, the Leibnizian framework remains out of reach by the precise quantitative definitions associated with measurable quantities in the usual physical methodologies. This is due to the infinite character articulating the different perspectives to each other through an endless recurrent sequence leading to different remarkable properties. Such remarkable properties, associated with one modality of measurement or another are here deduced and not postulated as usually done. On postulating them from the start (without any sufficient reason) one may examine how things occur in a certain way and not otherwise, but one cannot go to the root of their existence (the why) or to the order from which they result. With the usual methods, one deals with the "how" but never with the "why", reached (simultaneously with the "how") only through a formulation of the Leibnizian type. This is a direct consequence of the fact that the **usual models** are **not rooted in the necessary requirements** imposed by dynamics but **only with one of its modalities of existence** mathematically expressed through some remarkable pertinent property but nevertheless unnecessary. The absence of such a necessity is illustrated by the multiplicity of equivalent dynamical models relative to the same problem described under different parameters each of which associated with a remarkable property and a specific methodology. The above narrow definition of physics closes the door to the possibility of a higher rationality capable to encompass different scales and points of view simultaneously.

CONCLUSION.
Let us conclude by recalling that what seems to be impossible in the paradigm that rejects quality in favour of quantity, (asserting that "qualitative is poor quantitative") becomes possible in the Leibnizian framework, for which the quantitative postulates associated with the ratio property (velocity) or the additive requirement (rapidity) should not be given from the start, but discovered as solutions of some higher principle. In this regard, and in the restricted framework of Einstein's dynamics, it is what one may call a **"microscope function" $[D_\mu=(E/E_0)^{(a-\mu)}]$ that leads to an ordered multiplicity of points of view on motion. Such a function precedes the velocity ($\mu=a+2$) as well as the rapidity ($\mu=a$) and paves the way for their harmonious co-existence in a higher rational framework.** One includes thus, an infinite number of points of view on motion, although only four of them turn out to be practically useful, the others being more or less complicated combinations of the four basic ones. Let us also note that the number of points of view is deduced from the internal underlying mathematical structure, and not dealt with from the outside as usually done. This is rendered possible by the construction of a discrete combinatorial and formal framework absent from the usual formulations. This fact was suggested to me by a line of thought developed by Kurt Gödel, who spent many years working on Leibniz's methodology, defending it against the too restrictive paradigm of the current scientific enterprise initiated by Newton, adopted by Kant and pursued by the majority of physicists. Last but not least, it should be emphasized that **the present Leibnizian approach is not only fruitful in its exploration of the unknown rendering possible what seems impossible in a too restrictive frame (such as the generation of a quantitative dynamics out of a qualitative one through trans-subjectivity), but also in questioning what is supposed to be well-known such as the relevance of Huygens and Descartes dynamics with respect to modern ones**. These turn out to be partly explored on a rational ground not only because of the deficiency relative to the absence of a net distinction between different points of view but also because of the too narrow mathematical framework associated with presently available dynamical approaches operating in the restricted domain of continuous and analytical functions.

One should keep in mind that dynamics and particularly the dynamical concept of energy at its basis is also basic to physics as a whole, and not only linked to the elastic collision problem from which it is originated. Moreover, numerous works use an analogical procedure to pass from one discipline to another by keeping the same mathematics (same syntax) applied to another field of research (different semantics). In particular, the structure of Einstein's relativity theory was transferred to other frameworks (such as biology and information theory as shown by different authors among which Pinel for biology and Jumarie for information theory). Following the same line of thought, one may say that this dynamical Leibnizian methodology (that extends the Einsteinian framework in different directions), may apply to different scientific disciplines such as biology, information theory or still other domains, because of the universality attached to it.

The realms of necessity (essence) and freedom (modalities of existence).
The main reason for which Leibniz was misunderstood lies in the fact that no net distinction is made in conventional physics between the two orders of **necessity** (essence: conservation laws) and **freedom** (modalities of existence: points of view on motion**).** One basic point consists in noting that the requirement of two conservation laws constitutes an absolute necessity without

which the problem of elastic frontal collision (at the basis of dynamics) is unsolvable. According to Leibniz and contrary to popular wisdom as well as to conventional approaches of dynamics, the effective weight of this number "two" is much more than "one plus one". It can be used as a constraint that contributes to the determination of dynamics. Its use does not constitute a necessity since one is free to choose this constraint or another one as done by conventional physics. However, one should keep in mind that if such a constraint is not taken into account, this does not mean that it is not at work. Having recourse to non necessary postulates imposed from the outside, these add automatically to the necessary ones, leading to a rigid dynamical framework. Such rigidity (due to the intervention of postulates imported from an outside world foreign to dynamics, such as kinematics), can be avoided by imposing a hierarchical system of propositions, starting with the weakest necessary requirements. Such a hierarchy allows two main things: it leads to a better understanding of each step since the different constraints are imposed starting from the weakest postulates and getting progressively towards stronger ones. Then, the use of necessary requirements avoids the recourse to more or less arbitrary considerations (only partly justified), that interfere negatively with the necessary ones, leading to a too constrained dynamical system. This is typically the case with Newtonian dynamics (and, to a lesser degree, with Einsteinian one).

On postulating non necessary constraints, one mixes these consciously imposed ones with the unconscious ones present anyway in the dynamical structure, since they constitute its core: their absence signifies the absence of dynamics itself. In favouring freedom to necessity and in abolishing the hierarchy between these two distinct (although inter-correlated) orders one deals with physics in a purely empirical (Newton) or partially rational (Lagrange-Hamilton) way. A complete rationality requires the recourse to necessity allied to a certain hierarchy, allowing thus a better understanding of the different steps and leading to a smooth passage from "quality" to "quantity". In brief, one should emphasize the distinction between the two kinds of assertions: ***what one can do without necessity and what one should necessarily do***, *otherwise dynamics looses its very existenc*e. The different points of view on motion (velocity, rapidity, celerity, etc.) belong to the first kind of assertions, while energy and impulse belong to the second one.

Logical distinctions : "Double affirmations" versus "double negations".

As pointed out above, Achilles' heel of conventional physics is that it does not recognize the importance of the distinction between the realm of necessity and that of freedom. Logically speaking, it amounts to put a "double affirmation" and a "double negation" at the same level. In "usual rationality", the following assertions: "one **can deal** with motion rationally if one **introduces** the velocity concept" and "one **can deal** with dynamics rationally if one **introduces** conservation principles" are not, strictly speaking, true "double affirmations" apt to include other possibilities. These are to be understood as "double negations" for, according to the Lagrange-Hamilton formalism, at the basis of physics, this leads to: "one **cannot deal** with motion rationally if one **does not introduce** the velocity concept" and "one **cannot deal** with dynamics rationally if one **does not introduce** conservation principles". Indeed, until today, the usual teaching of dynamics perpetuates the concept of velocity as being the only way to deal with motion. This fact was not true in pre-Newtonian dynamics. The exact definition of the velocity concept was formalized by Varignon at the birth of the 18^{th} century. Such a fixed definition was included in the Lagrange-Hamilton formalism that brought a safe foundation of mechanics. Since this time, one may say that the velocity concept became the only candidate apt to deal with motion rationally. Any candidate such as the one that verifies $p = dE/dv$ becomes false or only true accidentally (valid only in the degenerate and local form of dynamics as proved earlier). This clearly shows that the usual rationality constitutes the basis of a **closed science** incapable to distinguish between what is **imposed by the situation** or the reality dealt with and what is **proposed by the mind** of the scientist exploring this reality; what amounts to the necessary requirements imposed by the object of study and what amounts to the relative freedom of the subject in proposing one conceptual framework or another. Conscious of the necessity of such a distinction that goes back to Aristotle (Essence and modalities of existence), Leibniz's philosophy of nature leads to the replacement of the above two "double negations" (associated with the conventional rationality) by one true "double affirmation" and one "double negation" as follows: "one **can deal** with motion rationally if one **introduces** the velocity concept" but "one **cannot deal** with dynamics rationally if one **does not introduce** conservation principles". The first assertion is to be understood "positively", in the sense that the introduction of velocity should not prevent us from proposing other different ways apt to account for motion according to various perspectives. Thus, in the Leibnizian framework, the "double affirmation" does not prevent automatically the imagination from investigating in the infinite realm of possibilities or potentialities. However, as emphasized by Leibniz – against Plato's free world of pure ideas, and in agreement with Aristotle's world of substance and its various manifestations – the freedom provided by the first "double affirmation" statement should be firmly held by a necessity through the "double negation" proposition, without which no predictive science is possible. It is the passage from Plato to Aristotle that the conventional physical rationality has missed. It is rooted in the realm of non necessary requirements (such as the one associated with the velocity concept that it considers as necessary as shown by practically all the teachings of Newtonian and Einsteinian dynamics). This lack of a distinction between what **"could"** be done and what **"should"** be done, actively contributed to the formation of generations of dogmatic researchers. In brief, one may say that the main problem with the usual interpretation of dynamics (which is at the basis of physical science), is that **conventional physics hastily replaces "If" by "If and only if ",** closing thus the door to any other degree of freedom than the one provided by the Lagrange-Hamilton formalism.

Common points and differences between Einstein and Leibniz methodologies.

Both Einstein and Leibniz relativities are based on two major principles. As well-known since the birth of special relativity, Einstein's principles are the **quantitative** principles of **kinematical relativity** and **invariance** of the light velocity. Leibniz's principles correspond to a **qualitative** principle of **dynamical relativity** (identity of indiscernibles) and a **quantitative** principle of **order** (plenitude). The first of these accounts for the idea of relativity in a qualitative way, independently of any specific point of view on motion, while the second one accounts for a quantitative order established between the different points of view, through a recurrent sequence allowing to pass from one scale to another generating thus different points of view (among which those derived from Einstein's relativity with its different versions and points of view).

Common points and differences between Lagrange-Hamilton and Leibniz methodologies.

Both Lagrange-Hamilton and Leibniz methodologies are based on a scalar concept having the dimension of energy; but Leibniz's scalar (called vis-viva, living force or absolute force) corresponds to a conserved entity, while the Lagrange-Hamilton scalar (called Lagrangian) does not satisfy any conservation property. Moreover, the two methodologies differ radically from one another: the least action principle at the basis of the Lagrange-Hamilton principle operates on a unique point of view (a simple curve) while the Leibnizian dynamical relativity principle (identity of indiscernibles) operates on different points of view simultaneously (a family of curves) among which the point of view associated with the Lagrange-Hamilton formalism. When the attention is focused on the velocity notion (usually attached to the least action principle), one discovers that each methodology presents its advantages and drawbacks. The Leibnizian methodology leads to a certain "economy of thought" since one encounters only three physical entities (needed in the resolution of a physical problem: energy, impulse and velocity) contrary to the Lagrangian which does not correspond to a physical entity. It corresponds neither to a conserved quantity nor to an element needed in the resolution of a physical problem. The Lagrangian is inherent to a specific methodology whose interest lies in the fact that it leads to a certain structural simplicity. This simplicity may be illustrated by the expression of impulse which is deduced from the Lagrangian through a simple derivative with respect to the velocity. It is worthy of notice that the association of the non-physical entity (the Lagrangian) with the non-physical operator (the derivative) leads to a physical entity (impulse). In the Leibnizian methodology, impulse is deduced from a physical entity (energy) to which one associates a physical operator (the translation operator). This physical operator is structurally more complicated than the usual derivative. In brief, one may say that when dealing with the Lagrange-Hamilton formalism, one encounters a "structural simplicity" associated with a "conceptual complexity" (lack of economy of thought), while when dealing with the Leibnizian methodology (restricted to the point of view associated with the velocity) one meets a "structural complexity" through the translation operator associated with a "conceptual simplicity" (economy of thought).

Thanks to the possible account of a multiplicity of points of view on motion, the Leibnizian methodology that leads in all cases to a "conceptual simplicity" (economy of thought), singles out one and only one point of view for which one has **also** a "structural simplicity". This point of view – called rapidity – is the one for which the physical translation operator coincides with the mathematical notion of a derivative.

Let us also note that, the present Leibnizian formulation allows getting a better understanding of the Lagrange-Hamilton formalism. The necessity of the Lagrangian appears clearly when we look for the solution of the principle of dynamical relativity, which corresponds to a second order differential equation relating energy to the velocity. Because of the **translation operator** which does not coincide with the **usual derivative** but with a deformed one associated with some multiplicative factor depending on energy, the second order differential equation becomes difficult to resolve. However, one shows that with an appropriate **change of variable** replacing energy by a new scalar, the second order differential equation greatly simplifies leading to a differential equation whose solution reduces to a closed curve (a circular form in non-dimensional notations as shown in a previous section) relating the velocity to the new scalar. This **new scalar** (resulting from the change of variable) corresponds precisely to the expression of the **Lagrangian**. To some extent, the least action principle reminds us of the old geocentric system of the world. It operates on a **circular form** which allows defining the physical entities – impulse and energy – in a simple manner. In addition, if one performs the same calculations associated with the other singular points of view, one discovers that only the point of view associated with the velocity notion leads to a closed curve.

Link to pre-Newtonian and post-Einsteinian dynamical frameworks.
Let us finally recall that the present work that deals with the principle of dynamical relativity following the lines of thought developed by Huygens and extended by Leibniz in two directions (exclusive multiplicity of worlds and inclusive multiplicity of points of view on each world), is of a double faceted nature. One of these concerns the future of dynamics while the other is related to its past. We have shown that, instead of looking at Einstein's dynamics as a deformation of Newton's one and at recent dynamical formulations as deformations of Einstein's one (as usually done), the problem may be dynamically tackled with in a general way from the start. This is obtained by considering the most general forms compatible with the idea of relativity associated with the constraints imposed by the requirement of conserved entities. These forms are cast into a generalized framework apt to admit irregular solutions. This allows showing the limit of validity of the conventional formulations as to their judgment of the first dynamical approach proposed by Descartes. In particular, it was shown that the Cartesian dynamical framework presents two non-negligible features. It is valid locally (like Newtonian dynamics) contrary to what is usually believed and it possesses potentially a rich structure (richer than that of Newtonian dynamics) obtained through a simple regularization procedure.

Let us close this work by noting that putting into question a well-established knowledge is at least as important as the exploration of the unknown. Let us also emphasize that any theory which does not show the basic concepts on which it is erected as well as the procedure by which it is produced constitutes a disguise and a mask behind which hides an authority ashamed to reveal itself as such.

POSTFACE

Leibniz's overlook on the structure of knowledge (relation to Kant, Gödel and Turing).

Leibniz was ahead of his time, looking for a "universal characteristics" apt to go beyond the frontiers of mechanism through the introduction of a systematic language or a specific formalism apt to conciliate different (apparently antagonistic) currents of thought. Scientific ideas associated with mathematical logic (at the basis of axiomatic physical approaches) are cast into a too narrow framework favouring analysis and formalism on intuition. Leibniz learned from Pascal the importance of distinguishing between two types of minds: the geometric and the fine or subtle one. Simple one-way rationality and analyticity remain too crude to account for dynamics in a subtle multi-viewed manner. With the advent of twentieth century science the research on the foundations of mathematics left the realm of geometry and intuition and followed a purely algebraic reasoning. Such a divorce between the geometric intuition and the purely formal and algorithmic procedures led however to main discoveries such as Gödel's incompleteness theorem. The removal from geometry is partly due to the disorder produced by the discovery of the different non-Euclidean geometries. Frege asserts that there is a form of delirium and disorder in the mathematics of the first half of the 19th century where new geometries are proposed. In order to correct this "delirium" and recover a certain unity and stability, the attention was focused on arithmetic (a logical theory "par excellence" for Frege). While mathematics and physics mutually fecundated each other since the renaissance, the science of logic, at the basis of the foundation of mathematics, remained autonomous and did not benefit from such a mutual fecundity.

Laplace (in physics) and Hilbert (in mathematics) carried the determinism as far as possible, while Poincaré (in physics: sensitivity to initial conditions illustrated by the double pendulum or the solar system) and Gödel (in his foundations of mathematics) showed the internal limitations of this paradigm. According to Laplace, the mathematical determination implies the predictability of the evolution of physical systems. For Hilbert, the formalization, at the basis of the determination of a mathematical theory implies that the formalized statements are decidable. **Poincaré** showed that certain deterministic physical systems are **non-predictable,** and **Gödel** showed that certain formal systems are **non-decidable**. These proofs rehabilitate the role played by intuition, violently and frontally attacked by Hilbert's formalism often related to Leibniz's desire of constructing a "universal characteristic" (apt to conciliate apparently antagonistic concepts). Unlike what is usually believed (particularly by physicists) Leibniz analytical investigations in different disciplines and his synthetic work belonging to metaphysics constitute a precious source of epistemological reflexions as shown below. It cannot be reduced to Hilbert's formal program.

In drawing the attention on the importance of formalism for production, efficiency and prediction (through his famous: let us calculate) Leibniz's methodology (possible existence of a universal characteristics) is in agreement with the line of thought developed by Hilbert. However, this is only one facet of Leibniz's philosophy, for he also adheres to a more profound qualitative level adopted in the 20th century by Poincaré, Thom and Gödel. Gödel's incompleteness theorem asserts that any formal system is **incomplete**, in the sense that its proper consistency cannot be demonstrated inside the system. This fact is somehow similar to the physicists view according to which dynamics is not auto-sufficient, in the sense that it needs an a priori external postulate to be consistent (velocity, rapidity, or any other point of view impossible to justify from the inside of dynamics). However, in addition to his incompleteness theorem, Gödel showed the possibility to enrich any formal system by new axioms compatible with the consistency of the original system. In dynamics, Lévy-Leblond showed the possibility of enriching dynamics by introducing different new points of view on motion, consistent with the original dynamical framework. This finite enrichment corresponds in Leibniz's approach to the construction of a hierarchical potentially infinite points of view, expressed through an infinite number of different formal systems. This **infinite procedure** circumvents the impossibility of dealing with dynamics without adopting an outside property or having recourse to outside information. The **external property** or outside information compatible with dynamics (but whose necessity cannot be demonstrated within the dynamical system) is here replaced by a **hierarchical principle** of order generating an infinite number of properties, some of which will turn out to be remarkable and singular. This leads to a more complete and systematic formalism encompassing the different dynamical models, leading to physical interpretations which are not pre-determined or postulated as usually done in dynamics. This form of completeness is obtained by an outside adjunction of a hierarchical principle (and not a single property) that cannot be demonstrated in the original system but which is nevertheless compatible with it.

Poincaré and Thom favour intelligibility to efficiency in their qualitative geometry or topology, where predictability is lost, while Laplace, Hilbert, Einstein and rational or analytical physicists (Lagrange-Hamilton formalism and metrical geometry) favour efficiency on intelligibility ensuring thus predictability. The return to a pre-Kantian, (pre-Newtonian) paradigm, and the rediscovery of Leibniz's thought with its two faceted nature associated with continuous systems (infinitesimal analysis) and discrete ones (combinatory analysis), allowed us to benefit from the two above-mentioned currents to construct a **predictive efficient dynamical framework** without sacrificing intelligibility and qualitative thinking. This is possible provided one distinguishes between "objective essence" and "subjective modalities of existence" as advocated by Leibniz and in a lesser degree by Aristotle before him. This distinction between an "**objective one"** and a "**subjective many**" [replaced in the Newtonian (or more generally the Kantian) paradigm by the sole existence and the conditions under which such an existence appears to us in some a priori space of configuration] is at the basis of the possibility of a **conciliation between intelligibility and efficiency,** and of a possible co-existence of objective quantities with subjective qualities. These are materialized through the so-called trans-subjectivity and inter-subjectivity procedures, impossible to form in the too narrow paradigm of the Newtonian or Kantian type.

The theoretical machinery associated with what we have called a **"microscope function**" allows one to pass from one scale (or point of view) to another in a repetitive endless way, **generating thus the properties usually imported from the outside**. Such a typically Leibnizian trick or ingenious way (relational by its very nature), allows to reach a higher unity that includes the different presently available dynamical models. How could Leibniz be so far ahead of his time? How could he conceive the inconceivable? The limits and borders do not frighten him, he crosses the frontiers to better observe them from the outside. The physicists have difficulties to delimitate what can be measured (by looking for ingenious ways of measurements each of which associated with a specific remarkable property whose existence should be as simple as possible). Leibniz's

optimism led him to the belief in the existence of a **higher principle** responsible for the generation of such singular and remarkable properties. This belief is, by nature, metaphysical and non-demonstrable inside any formal system. It is of an intuitive nature, and does not satisfy any absolute necessity just like the different properties associated with each point of view subject to a specific methodology. There is however a major difference between the two kinds of intuition. The first is somehow **substantial –** in the sense that it shares with substance its uniqueness and specificity – while the Leibnizian intuition is **relational**, founded on a **principle of order**. As Leibniz puts it (when criticizing the Newtonian and Cartesian mechanisms) it does not correspond to a **local and solid** constraint like a nail planted in a wall to which everything should be attached, but it corresponds to a **global and fragile** constraint, like a spider's web where everything is related to everything else in a multiplicity of ways and according to a certain specific order. If one breaks this order, everything collapses like a house castle. This belief in a hidden order (behind the dynamical remarkable and singular properties which is of no necessity) is certainly the reason for which Leibniz's dream of constructing a "universal characteristics" and his conception of knowledge and natural philosophy were considered as a pure fancy and illusion like building castles in the air. Such a structure has never been developed earlier, in the scientific era, because it does not come to the mind spontaneously but corresponds to an **ingenious construction,** motivated by the desire to go beyond oppositions favouring conciliation (but also avoiding any contradiction). Such a **non-contradictory conciliatory philosophy** does not impose itself; it is only required if one wishes to grasp a structure from all sides simultaneously (through a unified way). One may illustrate the Leibnizian modalities of existence through harmony like the one encountered in the construction of a song, where only one instrument is sufficient to inform us about its identity while the different instruments are needed to fully appreciate it. Each different dynamical model plays the same song with only one instrument, while the Leibnizian theory plays it with the different instruments echoing each other simultaneously. This obviously leads to a certain redundancy but it is this very redundancy that ensures a better intelligibility and harmony, lacking when any instrument is played alone.

Kant and Gödel.

Hilbert's program was to purge arithmetic from any substantial form by reducing it to a pure logical formalism. As known from the history of logic and philosophy, the first critic of this logical formalism took the form of a defence of Kant's philosophy according to which mathematics cannot be reduced to logic. Poincaré claims that even if the logical formalism turned out to be able to reduce arithmetic to logic, this does not demonstrate that mathematics do not require any intuition. Wittgenstein also used to say that there could be no thought whose truthfulness can be recognized by itself. All this sympathises with Kant's assertion according to which only phenomena can be attached to reason and not objects as such. To deal with dynamics, one needs an a priori intuition (such as a ratio space/time) to account for motion or any other a priori form. In Hilbert's formalism, the existence of an object is ensured by the non-contradiction of the conditions that defines it, so that one thing is either true or false since eternity. However, if everything is thought of as being either true or false, then the statements concerning the future are true or false at any time so that one may recognize them denying thus the contingency of events. This violently contrasts with the intuitionists, for whom the mathematical objects are **potential objects**. For them, as long as one proposition referring to a certain property is not demonstrated, this property is neither false nor true. Thus, there exists a state which is neither true nor false called "**non-demonstrable**". The existence of the object has no significance before being exhibited.

Gödel proved that there exists in any formal theory including arithmetic non-decidable propositions. There exists then an intermediate state which is neither true nor false as advocated by the intuitionists supporting the Kantian pure form of intuition. However, this intermediate state can be **demonstrated in a wider framework,** tipping the scales towards the formalist Hilbertian program. Let us also note that certain non-decidable propositions should be true for the consistency of the theory. In showing the existence of truthful properties impossible to establish by the formal system, Gödel brings a definitive distinction between "**truthfulness**" and "**demonstrability**". A proposition derivable from a formal system is true but the reciprocal does not hold anymore. There exist **true propositions** compatible with a formal system and **needed for its consistency** but which are **not demonstrable inside this system**. In dynamics, one may think of the additive property associated with the rapidity parameter. This property is needed to account for the consistency of dynamics through group theory, but it cannot be demonstrated inside the dynamical system. Being true but non-demonstrable, it corresponds to what Kant calls an a priori statement or judgment. This also holds for velocity and celerity, each associated with a specific methodology ensuring their consistency. However, these different a priori judgements become deducible in the Leibnizian conceptual system. This deduction results from the relational principle of plenitude (principle of order and auto-organization).

Leibniz and Turing.

Since dynamics is cast into a mathematical system, this highly formalized physical discipline distinguishes between "demonstrable" and "truthful" properties. The "truthful" ones are those postulated and needed to ensure the consistency of each rational framework. In the Leibnizian "multiple rationality" the truthful entities associated with remarkable properties are not postulated anymore but they become demonstrable inside the Leibnizian framework. Thus, the Leibnizian paradigm goes beyond the Kantian paradigm associated with the different dynamical rationalities. One of the main difficulties encountered by physicists to understand this situation is due to the fact that whatever the theory, one needs to postulate at least one truthful property (ratio or additive property…) which is not demonstrable inside the physical system. However, what Kant and the majority of physicists are not aware of is that one is not committed to postulate a specific **remarkable property** as usually done but a **hierarchical principle**: a generator of a multiplicity of different remarkable properties. In this regard, the **principle of plenitude** (principle of order) leading to **various remarkable properties** plays, in the Leibnizian framework, the role usually played by the **remarkable property** on which each dynamical model is based. The Leibnizian methodology recoursing to infinity leads to a higher level of formalization where each one of the Kantian a priori intuitions (velocity, celerity or rapidity) is weakened by the introduction of a formal principle apt to encompass them into a higher rationality. This can be done only if one finds a trick apt to extend the Kantian paradigm without destroying it. Instead of considering the different intuitions developed by Levy-Leblond (as shown before in this work) through **external, internal** and **mixed** procedures to deal with motion, one keeps only the formal properties common to each "intuition", illustrated mathematically through a remarkable property. Noting that the different intuitions differ for motion but coincide for the state of rest, one discovers that the state of rest corresponds to three points that coincide with each other (an accumulation point). It is then tempting to perform a general reasoning on this local fact by use of a mathematical procedure largely favoured by R. Thom and known as "analytic prolongation". Considering the **rest state** as an **accumulation point** (a dynamical intuition: a grain including potentially a tree) of an undetermined multiplicity (potentially infinite) instead of a **single one** leads naturally to a **treelike structure**. After the identification of the scale structure through the construction of the "microscope function" (that allows passing from one point of view to another in an iterative endless manner), one discovers that the three different points of view on motion (discussed by Lévy-Leblond) can be deduced and do not need to be postulated separately (each associated with a specific rational methodology) as usually done. This methodology was suggested to Leibniz by Pascal's distinction between the crude mind and the subtle one where the intuition associated with a single point is crude in comparison to the subtle consideration of this point as a multiplicity of points locally fused together. Notice that the intuition of an accumulation point as a grain potentially bearing a treelike form is structural and geometrical. It differs from the somehow substantial one associated with ether or space, in which motion is defined through a ratio between space over time.

This sort of intuition allows one to get not only beyond the philosophical Kantian paradigm that includes the different rationalities but also beyond its logical counterpart derived by Gödel. As shown in some philosophical writings on Kant and Gödel (www.insa-lyon.fr la recherche de la vérité universelle), Gödel's theorem corresponds to the transcendental Kantian truth. Kant decomposed the structure of knowledge into three parts: directly observable (sensible) knowledge, knowledge reached by reason (rational and analytical thinking) and transcendental knowledge (unattained by rational analysis). This decomposition has its mathematical counterpart through: knowledge by calculus (description), knowledge by demonstration (proposition derived from a formal system) and non-demonstrable knowledge (non-decidable propositions). In the same manner as Leibniz's introduction of the principle of plenitude recalled above allows to get beyond the Kantian paradigm, there exists in mathematics an analogy with the above described passage from a physical intuition to a higher formal structure, where the new intuition becomes relational instead of remaining somehow substantial. This analogy is provided by the logician A. Turing who distinguishes "intuition" from "ingeniousness" (the crude and the subtle). According to Turing, (Pour la science: les génies de la science N. 29 p.84) *the mathematical reasoning may be considered in a schematic way as a combination of two faculties called: intuition and ingeniousness. The activity of intuition consists in the production of spontaneous judgments which are not conscious chains of reasoning […]. The exercise of ingeniousness in mathematics consists in helping the intuition through adequate arrangement of propositions and possibly by geometrical figures or drawings. In pre-Gödelian times some scientists used to think […] that the necessity to recourse to intuition may be entirely eliminated. […] We have tried to see to what extent it was possible to eliminate intuition. We do not care for the knowledge of what quantity of ingeniousness is required and we make the hypothesis that it is available in an unlimited quantity.* Turing's desire was to push to the extreme, the limits of the mechanical construction. He looks not only to circumvent Gödel's theorem but also to show how the concept of non-decidability may be somehow rendered **relative** rather than **absolute**: instead of considering it as some unsurpassable absolute he makes of it a concept **relative to each logical system of an unlimited hierarchy.**

In brief, with Gödel's theorems one is allowed to pose insoluble questions since their answer depends on the subject providing the solution. Gödel clarifies the notion of indemonstrable truth, purging the formalism from its pretension to absolute truth looked for by Hilbert and by many philosophers since antiquity. Gödel rehabilitates thus the role of intuition already introduced by Kant (through the introduction of his synthetic a priori and pure form of intuition), in his opposition to classical rationalism (Descartes, Leibniz, Malebranche, Spinoza …).

Let us also notice the analogy between Leibniz's "subtlety" and Turing's "ingeniousness". Leibniz's idea of "subtlety", (linked to a principle of order leading to a treelike structure possessing an unlimited number of branches), transforms the Kantian absolute character of an priori form of intuition, into a relative one associated with each branch corresponding to a specific model. In the same way, Turing's introduction of "ingeniousness", transforms the Gödelian absolute character of a non-decidable proposition into a relative one linked to each logical system of an unlimited hierarchy.

Leibnizian philosophical inquiry on perspectives compared to the usual progress of science.

In reason of the cumulative character of the scientific enterprise, it is rare that one uses philosophy and history of science to bring new elements apt for a renewal of a given discipline. This cumulative character appears in the titles of the research programs and the papers associated with them: empirical research often starts from the current "physical truth" and tries to deform it in order to include other elements. This is, for instance, the way traced in the passage from Newtonian dynamics to that of Lorentz-Poincaré and Einstein, as well as to deformed special relativity where the Poincaré group is deformed according to certain criteria [7-8]. *When the attention is focused on the coupled effect, then scientists use the term "doubly" instead of "deformed" to show that the new approach includes not only one coupling parameter (light velocity) but two, adding thus a specific fixed energy such as the Planck's energy.*

Another type of research, closer to rational formulations, is devoted mainly to the foundation, on a firm ground, of a given discipline absent in the process of discovery and lacking in the first attempts and investigations. In this framework one encounters typical titles such as "a new derivation of Lorentz transformations". Since these transformations are at the basis of modern physical science, these are not put into question but they are justified through different manners. This second type of research is of a doubly faceted nature: a positive and a negative one.

On finding the appropriate language and logic that lies behind such structures one gains into intelligibility, but one cannot grasp anymore some of the corners and corrugated areas that may be suggested by other interpretations and different investigations: all has been smoothed and polished. Once a point of view has been adopted through a certain relation (velocity for Lorentz and Einstein) any other relation becomes either false or a simple consequence of the first one. This purification process is not only positive (in spite of the clarity and precision that it brings), since it selects one conceptual perspective at the expense of other possible basic concepts.

The present Leibnizian **inclusive framework,** apt to include a **multiplicity of points of view** on motion, provides a precious indication as to how most of these papers and research programs associated with the "usual rationality" remain confined to the same exclusive logic, since this

exclusive character is never put into question. The adoption of another possible perspective: the "emergent rationality", operates with the same exclusive logic and does not address the question of the possible existence of a higher framework apt to include both rationalities and adding still other points of view. **The idea of an infinite multiplicity of points of view, anchored in intuition, transcends any physical measurement**. It comes to the mind only if one considers it seriously, from the start, as a basic entity of physics, and not as a reaction to difficulties in the usual adopted view. In such a case, one performs the necessary arrangement to counter the difficulty without any critical or deep thinking about the structure of a physical theory. This is the main reason for which the present approach cannot follow the usual procedures. The rupture is due to the fact that the starting point is anchored in qualitative considerations whose potentialities are infinite. This violently contrasts with the usual strong physical requirements based on a quantitative definition of motion.

Interest of ideas of the past linked with the origin of a subject matter.

This paragraph aims at showing how rich and fruitful is Leibniz's methodology, and its interest, compared to Einstein's one or to presently available formulations. At first sight, such an assertion appears paradoxical, for it seems to contradict the idea of progress. However, a deeper examination shows three major elements that constitute solid justifications of this unusual affirmation. The first of these is philosophical, the two others are structural. The philosophical point is that the scientific reductionism, initiated by Newton and followed by the scientific community, applies to Einstein and his followers but not to Leibniz, still in contact with the old Aristotelian paradigm where motion is not yet reduced to a simple locomotion or transport in space. This keeps the door open to possible different and multiple modalities of existence, historically rejected by the imposition of the unique scientifically recognized modality associated with the velocity concept. It is clear that, if one stops at this qualitative level, one remains unable to grasp the quantitative reasons essential to the construction of a predictive dynamical framework. (Moreover, Leibniz is not the only philosopher who criticized the mechanistic reductionism in the 17^{th} century).

To enter into quantitative thinking, one should recall two structural and correlated main points. The first one is general, the other is specific to Leibniz. It is important to keep in mind that the structure of Einstein's dynamics does not require any sophisticated formalism (absent from 17^{th} century science) such as those developed in twentieth century mathematical-physics. Einstein's dynamics (hyperbolic character) is embedded in conics that deal basically with parabolas, hyperbolas, circles and ellipses. Nowadays, it is well-known that, in so far as the relation between conserved entities is concerned, the closed circular and elliptical curves are excluded from dynamics, because they violate the causality principle. Thus, one is left with a parabolic Newtonian dynamics and a hyperbolic Einsteinian one. In addition to the fact that the mathematical structure of dynamics is included in conics (whose study goes back to antiquity), one should emphasize the renewal of these studies analytically and not geometrically as before the 17^{th} century. Through the development of the Leibnizian differential calculus, Leibniz championed this view by his discovery of the explanation of the "finite" by the "infinite". More precisely, in addition to Descartes proposition of extending the polynomial equation of the first few orders to higher orders of any rank called "algebraic equations", **Leibniz discovered that as long as one does not pass from the arbitrarily high but nevertheless finite order to an**

infinite one, no explanation is possible of some elementary facts. In particular, he discovered that the analytical expression of π which may represent the area of a unit disc (finite) may be dealt with exactly provided one expresses it in the form of an infinite polynomial development as follows : $\pi/4 = 1 – 1/3 + 1/5 – 1/7$ ….. instead of the never ending disordered form $\pi = 3,1459$…. According to Leibniz own terms, **simplicity requires ordered infinity** while usually infinity was then associated with divinity transcending any human reach. Leibniz brought thus “transcendence” from heaven to earth, calling the expressions whose polynomial development is infinite “transcendant expressions”. The discovery of the above mentioned infinite development required working with the following expression $x = \arctan X = \int dX/(1+X^2) = X – X^3/3 + X^5/5 – X^7/7$ ….. that reduces to

$\pi/4 = 1 – 1/3 + 1/5 – 1/7$ when $X = 1$.

In his study of the catenary’s curve Leibniz encountered: $x = A \operatorname{arcsinh}X = A \int dX/(1+X^2)^{1/2}$. He also discovered that Huygens’s solution adopting a different point of view corresponds to $y = A \arctan X= A \int dX/(1+X^2)$. More generally, he showed that all these are intimately linked with the hyperbolic structure that add to the trivial parameterization $z = A X$ (proportionality relation) that Leibniz used to write as follows: $z = A \int dX$ (in order to distinguish between what he called a “perception” dX and an “apperception” $X = \int dX$) . Thus, Leibniz could easily discover that the three points of view may be expressed in a compact form as follows: $x_\mu = A \int dX/(1+X^2)^{\mu/2}$ where the three parameters z, x and y correspond to x_0, x_1 and x_2 so that a certain order is here at work. The important thing that deserves to be emphasized here is that when the modern physicist deals with the hyperbolic Einstein’s dynamics according to one or another point of view, he uses relations such as $u = \sinh w$, $U = \sinh W$ as well as $v = \tanh w$ and $V = \tanh W$. The use of functional relations associated with $w + W$ (for translation) leads to composition laws of the following forms:

$$u \bullet U = u[1 + U^2]^{1/2} + U[1 + u^2]^{1/2} \text{ and } v * V= [v + V]/[1 + vV].$$

When dealing with these considerations by use of properties (not yet developed in the 17th century) and tables developed later on (in the 18th and 19th centuries) one is far from the origin of these different parameterizations intimately related to integral forms. Here lies the main reason for which Leibniz was able to immediately see what is less obviously seen today. If the above mentioned integral properties are known since Leibniz’s discovery of differential and integral calculus, these were the only information that Leibniz had, while today the multiplicity of forms makes of these relations a minute part of a huge whole. As well-known: **too much information kills information. Leibniz had the necessary and sufficient information to discover a certain order governing an infinite multiplicity of points of view on the hyperbolic structure, while today one should make a severe selection to single out these properties among a great number of other ones.** This tends to show that a fruitful idea is out of time: in other words, **unlike our biological material that wrinkles in getting old, a good idea has no wrinkles**. However, in the same manner as heart transplantation or skin grafting are rejected by the body, if one does not take all the necessary dispositions and operate carefully, the transportation of an idea from one mind to another requires the same care, otherwise it is rejected. It is only when this is achieved properly and with details that one is able to discover some missed ideas of the past among which the basic idea of multiplicity of points of view on a given reality. One may say that the mind, like the body, has this tendency to reject any foreign entity to which he is not accustomed.

<u>Multiplicity of Leibniz centres of interest; the need for the "Vinculum" hypothesis.</u>

In addition to the multiplicity of points of view on a given reality, there is another multiplicity associated with the numerous centres of interest and different disciplines in which Leibniz was involved. As pointed out by Leibniz himself, one may transfer a reasoning from one discipline to a totally different one, semantically unrelated to it. In spite of such a strict separation, the same internal logic may operate in both disciplines. In order to really grasp what underlines Leibniz's way of thinking, and particularly the different references he makes to various apparently unrelated questions, one should study the different subjects that appeared particularly interesting to Leibniz such as mathematics [9], theology (correspondence with Des Bosses) [38] and natural philosophy through dynamics [1-6] and his correspondence with Clarke (Newton's disciple). In Ref.[39], devoted to Leibniz's metaphysics, one reads : "*he (Leibniz) considers the question about the relation between the mind and body in a corporeal substance to be similar to the one concerning the unification of the divine and human natures of Christ. That is, he takes both questions to reduce to the same thing**: "how can two things with different natures be unified in one substance"?*** Thus, one should not content oneself with one discipline in order to grasp what Leibniz means by some of his assertions. To give but one example of this fact, let us recall that the main intuition associated with the link between unity and multiplicity, through the fact that a point (state of rest) is regarded as an accumulation point rather than a simple one, was encountered in a text dealing with theology and more particularly with the problem of "trans-substantiation" associated with a thesis [41] entitled: "le Vinculum substantiale chez Leibniz". According to Leibniz (p. 14), the essence of a "body" is to be associated with motion. The author adds that Leibniz was convinced that the true rules of motion are not as they appear to us and that in most beings **what appears to be "one" is "many"**. The present approach confirms these considerations where each one of the two conserved entities (energy and impulse) is precisely "many" in the sense that it may be accounted for through different points of view. The present work also shows that the essence of substance lies in its state of motion accounted for through an infinite number of ways that become indiscernible and fused together for slow motion.

In the same thesis p. 25, one reads that the monads enter into the composite not as "ingredients" but as "requisites", and not in virtue of a metaphysical necessity but of a simple **physical requirement**. This fact is essential since there is no metaphysical necessity associated with the existence of a multiplicity of monads. One monad may be sufficient to represent dynamical reality according to a certain point of view. This explains the reason why physical formulations do not necessarily need the idea of a multiplicity of points of view. However, at some scale of a reality, one useful perspective (or "monad") may loose its operational character, becoming useless in some respects. That is what happened to the concept of velocity in high energy physics as shown before. Thus, the multiplicity of monads is a practical and physical requirement, each of which being useful for some tasks and useless for others. It is neither metaphysical nor a logical requirement. The asymptotical behaviour of the velocity concept does not suffer any logical or mathematical discrepancy, but only a **physical** one: it simply suffers from the impossible physical distinction between the curve and its asymptote.

In page 23 of Ref.[38] C. Frémont writes that if the introduction of the "Vinculum substantiale" served initially to deal with the theological problem of trans-substantiation, it mainly allows the **construction of the notion of a composite substance without associating it to an aggregate**. This initially theological question opens on a philosophical one: How to pass from simple substances to the unity of composite ones? The author proposes then to read the answers that Leibniz addressed to Des Bosses concerning trans-substantiation as a full text endowed with its internal logic independent of the circumstances for which it was written. All the characteristics of the "Vinculum substantiale" are intimately linked to this sole constraint: **nothing should be added to the composite by the "Vinculum" except its transformation to a true unity**. The "Vinculum" does not concern any modality of existence as such but it provides them with a better foundation**. The "Vinculum" is what realises the unity of a composite body:** it is then a principle of action. (The principle of action in the monad that constitutes a particular point of view is called a perception. The Vinculum should be active without being associated with a perception that characterizes the monad). This mode of activity, differing radically from that of a perception, is, strictly speaking, **not a being but a relation**. Its relational nature makes of it an element of a higher order able to produce a true unity without adding anything to the monads except harmony, by providing the reason of their existence and their ordering in one way rather than in another. **This higher unity does not only govern the universe but it constructs it**. Being simply "requisites" and not "essences", the monads can exist without reference to the Vinculum that ensures their order and the reason of their existence without any modification as to their physical manifestations.

These general considerations concerning the "Vinculum substantiale" are in a complete concordance with the present dynamical approach, the principle of action associated with the "Vinculum" agrees with what corresponds to objectivity through "inter-subjectivity". We have been able to produce efficient modalities of existence or isolated "monads" without recognizing in them the "monadic" character, telling us that the constructed entities are but a part of a certain whole yet unrecognized as such : its tree-like form was not yet discovered. The term "monad" is legitimated only when recognizing that behind the constructed models lies a unity composed of an infinite multiplicity of entities complementing each other and constituting the furniture of the visible world.

A fundamental difference between Descartes, Leibniz and Kant.

Unlike the mystics, whose main wish is to meet (or to see) God and benefit individually of such a sight or meeting, Leibniz's wish is to see **in** God or more precisely to see the **transmissible** plan God made in such a way that each thing is at a specific place and not at another location. This transmissibility is not possible in Descartes view for which God's plan transcends human capabilities of grasping it physically and intellectually. For Leibniz, only the physical part is humanly unreachable, since the account for an infinite number of points of view is physically impossible, but there is no essential difference (in principle) between God's mind and the human's mind. On placing God's intellect at the same level as the human intellect, Leibniz created a link and erected a bridge between divinity and humanity, where no intrinsic intellectual difference is at work. The difference lies at the levels of will and power. If God is infinitely more willing and powerful than man, so that an abyss separates the divine will and power from the human ones, there is no such an abyss separating divinity from humanity, in so far as the intellect is concerned. These opposite Cartesian and Leibnizian conceptions of the relation between divinity and humanity have their counterpart in the history of philosophy through the so-called “univocal”(Leibniz) and “equivocal” (Descartes and Kant) characters of the supreme being. In particular, when Kant attacked metaphysics and theology as to their uselessness in the study of natural phenomena, his attack is only valid against Descartes. If God’s intellect is unreachable as claimed by Descartes, then the recourse to God contradicts rational thinking: no relation is possible with God’s intellect and hence God’s plan. Thus, Kant’s philosophy constitutes a relevant answer to Descartes useless recourse to God in his justification of conservation laws. However**, Kant does not provide a decisive answer against Leibniz’s recognition of the existence of an active principle behind the contingent manifestations**. Leibniz’s belief in the existence of a higher principle governing the different modalities of existence could not be countered, for what is planned by God may be reached by a human being since the same rationality is at work in a divine mind as well as in a human one. The difference is not of nature but only of degree, so that what a divine mind grasps instantly requires a long time for humanity to seize it. This is the main reason for which Leibniz contributed actively in the creation of scientific institutions so that the human spirit reaches collectively and across centuries a better understanding of the fundamental bricks of nature and the cornerstones of the knowledge of physical reality. Since Kant’s distinction between what is knowable (science) and what is unknowable (metaphysics), most physicists rejected Leibniz’s methodology, considered to belong to an unknowable realm, and hence unworthy of scientific investigation. But such a decision is not rationally rooted as shown in this work. From a logical standpoint, it should be remarked that one cannot prove the inexistence of a higher principle that transcends the contingent ones. One can only postulate such an affirmation. On following Kant’s epistemology concerning the necessity of discarding Aristotle’s and Leibniz’s metaphysics from the realm of science in favour of the Newtonian paradigm, most of **the physicists confuse what science decided not to know with what is intrinsically unknowable.** The present work shows, beyond any doubt, the existence of a higher methodology allowing to unite the different presently available dynamical methods into a unique framework provided one makes the necessary distinctions (objectivity, subjectivity, trans-subjectivity, inter-subjectivity, infinite multiplicity of points of view, possible worlds etc.) and associated principles (sufficient reason, plenitude, relativity, indiscernible etc.) absent from conventional approaches. Having ascertained, along its history, that the different philosophical systems that distinguish between “transcendence” and “contingency” were not fruitful to the scientific enterprise and to the discovery of natural laws, the physicists deduced that this distinction is purely metaphysical and has no existence in physics. However, this is not a proof but only a conjecture that turned out to be false as shown in this work. To prove the importance of such a distinction and hence the falsity of the above-mentioned physical conjecture, it is sufficient to actualize the richness of the distinction between “transcendence” and “contingency”, showing its relevance to physics. The possible realisation of the Leibnizian conciliatory attitude through the effective discovery of a higher unity in dynamics was a great surprise to me. The more a thing is surprising the more it seems important, because of the huge distance that separates the initial belief in a conjecture anchored in the minds and the process of discovery which may turn things upside down. Many examples are provided by different physical and mathematical discoveries showing that **the importance of a structural or conceptual work is measured by the intensity of the surprise provoked by its actualization**.

Let us go back to Leibniz and emphasize that the continuous principle that links God to man (as well as to other subtler creatures – angels and prophets – and cruder ones – animals and plants–) considered as more or less capable of light reflection as compared to the bright light emanated from God's intellect has been countered by the religious authorities, for whom the transcendence of God is absolute and placed beyond any human reach. ***Leibniz wants to save God from the tyranny of all kinds of unjustified religious creeds***. In a sense, one may say that **Leibniz position was uncomfortable, attacked from both sides: the realistic physical one, anchored in earthly visible (measurable) beings, as well as the idealistic religious one anchored in the heavenly invisible entities**. In modern non theological terms, Leibniz is looking for a higher principle that allows one to see why one point of view is articulated to another one in a way rather than in another. This constitutes the core of the "principle of sufficient reason" rooted in Leibniz's belief in the capacity of the human mind to reach the ultimate causes, usually attributed to divinity. **This makes of Leibniz a heretical individual from the standpoint of religious authorities and an idealist dreamer from the standpoint of the scientific institution** (anchoring science in special modalities of existence). Until recently, the scientific institution could not believe in the existence of a higher unity that transcends and explains the different points of view adopted on a given reality. The very idea of different points of view on motion was not recognized before the works of Wheeler, Lévy-Leblond and Comte followed recently by many others, although this fact did not yet reach the general teaching. If the principle of sufficient reason goes back to antiquity, it has never been developed seriously enough, except by Leibniz who made of it the root of his multiple-rationality. This principle is rarely adopted in physical science and when accounted for, it remains associated with a **regional** vision linked to a specific point of view or another. But the main interest of this principle is to go beyond points of view in order to explain the reason for which the different points of view occupy one location rather than another. This is not possible as long as one performs arbitrary choices such as u, v and w. Only the introduction of an "inclusive logical framework" starting with v_μ allows one to deal with dynamics in an ordered way operating on the infinite multiplicity of points of view through the Greek index μ as shown in this work. Let us finally emphasize that Leibniz is a man whose attention was attracted by the relativity of things and the finiteness of each being. As pointed out by Deleuze in Ref.[42], concerning Leibniz philosophy : *"the organism is defined by its capacity of folding its parts to infinity and to unfold them, not to infinity, but to the degree of development assigned to each specie"(my translation).*

Even the most powerful and intelligent man moves heavily on the ground so that what he can see remains local and very partial, while a simple bird is capable to move from one hill to the other capable thus to encompass what no man can do naturally. In a word, Leibniz's methodology is likened to a bird capable to fly from one branch to the other in the tree of knowledge. His critics of usual analytical methodologies can be metaphorically likened to worms creeping on branches (Einstein's dynamics in its two versions: "usual and emergent rationalities") or on the trunk (Newtonian dynamics where both rationalities coincide). The worm is unable to see that the piece of wood on which it creeps belongs to a more complex figure (tree-like structure) nurtured by the sap coming up through the roots and trunk grounded in a fertile soil. **The figure of the "angel" constructed by the human mind is probably one of these figures rooted in man's desire to be a bird**. As the saying goes: **"Even when a bird walks one feels that it has wings"**. This marvellously applies to Leibniz's conciliatory methodology. When Leibniz evokes reality, his assertions are directed towards peace and conciliation without sacrificing consistency. He was looking for a higher principle that transcends physical contingency. Unlike most of us, walkers, crawlers and creepers, Leibniz believed that man is capable of reaching the heights and ultimate causes usually attributed to divinity or to a supreme being. His desire to reach a profound understanding of the possible links between various elements, led him to put deep in earth the grain conveyed to become a fruitful tree. The more the roots are deeply buried in the ground and nurtured by a fertile soil, the more the tree is capable to erect and elevate in the sky. The metaphor of the grain becoming a tree that goes back to Aristotle played a major role in my different investigations, not only from a metaphorical standpoint but also at a structural level where the different points of view converge towards a fixed point according to a unique tangent constituting the trunk of a regular tree-like structure.

The principle of continuity hidden behind Leibniz conciliatory attitude.

This paragraph has been suggested to me by the work of C. Frémont in Ref.[38].

According to Leibniz, there is a violent lecture that does not encourage conciliation, complementarities, and a construction of a harmonious world where all the minds unite in view of a better comprehension of the physical as well as the moral or ethical world in which we live. Leibniz confesses his doubts as to the usefulness of the multiplicity of authors and works that are opposed to each other without any common purpose. He even feared that men would return to a barbaric age because of the huge multiplicity of books and authors (that will fall into oblivion in a few time)**. How to speak of a Republic of minds in Europe if to a maximum of means correspond a minimum of effect?** Leibniz is the only one who proposes a peaceful lecture of history rather than a lecture favouring ruptures and oppositions. He writes to Des Bosses that his contemporaries could not conciliate the ancient with the new philosophy because of a bad lecture of the Aristotelians. For Leibniz, a bad lecture is firstly one that does not take into account the conditions in which the concepts are valid transporting them inconsiderably to other questions. The notion of "entelechy" is good to metaphysics but useless in the practice of physics. The concept of "atom" is appropriate for the description of corporeal phenomena but not for their explanation. Secondly, a bad lecture is one that proceeds by negation more than by affirmations. Leibniz writes: "I found that most of the Sects are right to a great extent in what they propose, but not in what they deny". One should draw the quintessence of the best writings by a voluntary positive lecture, in order to select the hidden truth buried among an infinite multiplicity of superficial propositions. **According to Leibniz, truth is more prevalent than what one thinks, but it is weakened, mutilated and corrupted by superficial additions that render it useless**. The history of philosophy is "monadologic", in the sense that it should be written according to the law of harmony: each sect or doctrine may be characterized by just one point of view. It is not only ignorance that may produce a return to a barbarian age. The horrible mass of opposite books and authors, each pretending to hold the truth as well as an authentic knowledge, would lead to the same result because of the confusion it produces in the minds. Both contribute to the cancellation of the ordered relations through common points and differentiations that constitute what one may call a culture. **One should be able to distinguish between the circulation of basic ideas and the huge amount of poor information that risks erasing, by inundation, the pertinent traced ways.**

Leibniz attachment to Aristotle and his refusal to innovate the whole of philosophy as wished by Descartes is well exemplified by the following assertion: "I do not pretend to be an innovator; on the contrary, I find ordinarily that the most ancient opinions and the most recognized ones are the best". At first sight, and as advocated by a number of authors, one may find that Leibniz's conservatism is dangerous to scientific thought which has been developed by ruptures and revolutions. However, one should recognize the existence of some basic principles and fundamental distinctions difficult to put into question. This is the case of the distinction between "essence" and "modalities of existence" initiated by Aristotle, whose importance has been shown in this work. Considered by conventional science as a metaphysical assertion with no use in dynamics, this framework became the object of a great number of controversies, misunderstandings leading to a general epistemological disorder. This clearly underlines the interest of deepening and widening one's conceptual framework before pretending to have said the last word on motion by simply verifying that the theoretical assertions fit some experimental data as done since the first scientific revolution that occurred in the 17th century. These assertions constitute a rational justification of Leibniz's conciliatory attitude. His basic philosophy is never to cut, to censure, to condemn, to suppress, but to welcome and to look for the ultimate reasons behind each proposed philosophical system as well as for the common points between the different proposals. **A non violent lecture is one that knows how to welcome and reform at the same time**, otherwise the conciliation becomes artificial and inconsistent leading to no progress of knowledge. On the contrary, such a conciliatory attitude with no regard to coherence would lead to obscurantism, confusion and therefore one falls again in Barbary which is more serious than the one that results from ignorance. If a polemical discourse constitutes a bad guide, a superficial conciliation that does not account for the principle of non contradiction is worse since it leads to confusion and error. One may refer to I. Stengers (ecology of practices) as well as to M. Serres and C. Frémont for more details on this subject matter.

The present work : a typical example of a Leibnizian non violent reading of physics

Leibniz looks for **a true conciliation requiring a net distinction between scales and points of view**. The present work clearly shows why, contrary to appearances, Descartes and Huygens may be conciliated, and what are the conditions of such conciliation. Both scales and points of view are necessary for such conciliation. If one considers the scale that corresponds to the 17th century experiments, then no conciliation is possible: at this scale Descartes proposition is false as attested by empirical scientists. Descartes dynamics is valid for high energies while Huygens's one applies only for low energies. In addition, if one does not take into consideration the fact that motion can be dealt with according to different points of view, then the conciliation of Descartes and Huygens becomes again impossible, for each one favours one particular point of view even if none of them was conscious of this. Structurally speaking, only in dealing with tree-like structures (families of curves) one is able to obtain such distinctions. As long as one deals with branches (isolated curves) each constituting one point of view (the velocity in mechanics as taught in elementary courses and rapidity in more advanced ones), there is no way to conciliate Huygens with Descartes. Worse, in this too narrow framework, and in comparison to Newtonian dynamics, Huygens's formulation appears to be quantitatively true, but conceptually false because of the local character of Newtonian dynamics; while Cartesian dynamics appears to be quantitatively and conceptually false. This conclusion is a direct result of the narrow character of Newtonian space-time physics that considers neither scale effects nor any multiplicity of points of view on motion.

The problem is not resolved in the more extended Einsteinian framework. If this framework is wider than that of Newton, it remains, nevertheless, too narrow because of its association of motion to the Lagrange-Hamilton formalism that selects a **unique point of view** corresponding to the notion of velocity (through its relation to the Hamilton first canonical equation $v = dE/dp$). In the case of Einsteinian dynamics, Huygens procedure that corresponds to $p = dE/dv$ and Descartes one $E = m|v|$ are both quantitatively and conceptually false. However if one admits to interpret these dynamics associating with each of them one different point of view $p = dE/dw$ (Huygens) and $E = m|u|$ (Descartes), where the two points of view w and u differ from each other as well as from v, then Huygens dynamics and Descartes one acquire a certain coherence and are not to be rejected anymore. This broadmindedness initiated by Leibniz is the only way that leads to a possible solution but the price to be paid is the adoption of an inclusive logical framework that encompasses the usual one adopted by the majority of physicists. Such an adoption requires the distinction between "essence" and "modalities of existence" where v, u and w correspond to different complementary ways of measuring motion. This is a significant example that shows, beyond any doubt, the richness of Leibniz conciliatory attitude that has been despised for centuries by most rational physicists, whose short sighted view continues to work havoc in the scientific community. (A serious study deserves to be performed in order to show the number of valuable past and present young scientists that leave the scientific enterprise because of the brutishness generated by the dogmatic creeds of some influential narrow-minded individuals).

Before closing this somehow technical parenthesis, let us note that it is a contingent fact that Descartes dynamics as well as Huygens's one produce a physical example in agreement with Leibniz's conciliatory attitude. One may easily imagine other dynamics which are simply irreconcilable because one of them or both violate the principle of relativity. The reconciliation is only possible with different formulations, provided that each one verifies the constraint imposed by relativity. Without the existence of a common substrate that two different propositions verify, there is no possible reconciliation. Here lies the coherence of the conciliatory Leibnizian formulation. If one chooses arbitrarily some structures rather than others, one is practically sure that the conciliation is impossible, except by a chance which corresponds to an extraordinary and extremely rare circumstance.

It should be emphasized that **Leibniz's lecture of ancient and modern authors has the merit to welcome and reform at the same time**. This makes of the Leibnizian conciliation a true philosophical one, far from those conciliations that simply gather irreducible structures impossible to link to one basic substrate encompassing both of them with no contradiction. In Leibniz's sense, **conciliation is being capable to see behind the different doctrines the weakness of each of them as well as their potentialities**. In discarding their weakness and benefiting from their potentialities, one is then able to find out a possible framework apt to include both of them. Here lies the supremacy of Leibniz conciliatory methodology and its usefulness to a non dogmatic and non violent positive lecture of science.

<u>Difficulty to escape from current ideas. Discovery of a higher understanding of physics.</u>

In Science there is not only well-established research programs and conjectures and hypotheses that deserve to be invalidated or affirmed through proofs and demonstrations. There are also ideas, theses and proposals of the past, well-known for being without interest and deserving to be buried for ever and rejected from the realm of rational thinking. If differential calculus and the letters exchanged between Clarke and Leibniz concerning the concepts of space and time are considered to be of some interest to mathematics and physics, the monadology and the letters exchanged between Des Bosses and Leibniz on the mystery of trans-substantiation do not seem to deserve more attention than Newton's writings about alchemy. It is then very difficult to convince rational scientists that the ideas developed in Leibniz metaphysical writings are not only of some interest but they may shed light on some past and present physical problems, such as prenewtonian dynamics and posteinsteinian ones as shown in this work. **One of the major difficulties in physics is not to produce new ideas but to escape the current ones whose ramifications occupy all the corners and recesses of our brain.** What is already realized is certainly less important to a researcher than what remains to be achieved, but the solution of the unachieved problem is not necessarily due to the developments of new tools of analysis and invention of new mathematics as usually believed; some solutions may be hidden in the missed occasions where the problem was only looked at from only one angle of vision. **Each time one achieves a tiny section of a great idea that turns out to be fruitful on an empirical ground, the joy of the discovery and the enthusiasm generated by its practical applications diverts the attention from the conceptual level keeping the major part of this idea unachieved.** This is precisely what happened through the development of Newtonian dynamics where the relativity principle (developed scientifically by Huygens) remained dormant for centuries until its rehabilitation in the 20th century and only in specific cases. In particular, and from a structural standpoint, if the notion of a derivative plays an important role in physics, one should recognize that the use of this notion by Newton is totally different from its use by Leibniz. For Newton, the derivative is merely a compact notation associated with a purely descriptive feature of motion distinguishing the instantaneous velocity from the mean one (time derivative of a position space). It also occurs in dynamics through the definition of the force as the time derivative of impulse. For Leibniz, the derivative is the generator of the conserved entities which constitute the essence of dynamics without which this science looses its very existence. Thus, the same mathematical tool may be considered as superficial and descriptive in mechanics (Newton) or essential and generative of the basic form of dynamics (Leibniz). It should be recalled that, in spite of the development of rational mechanics by Lagrange and Hamilton where the derivative plays a major role (Noether's theorem), its use remains different from that of Leibniz. This is due to the fact that Lagrange follows Newton adopting his conceptual framework rendering it rational, while Leibniz follows Huygens whose conceptual framework differs from that of Newton (as shown in this work). In particular, as long as one reduces rationality to the one developed by Lagrange and Hamilton, one has to admit that in such a narrow conception of rationality the Leibnizian introduction of the derivative as a generator of conservation laws appears to be valid only accidentally in the Newtonian parabolic framework. Only recently, this procedure has been rehabilitated and considered to be much deeper than what was initially believed. **The genius of Leibniz is not only in finding out a new way to apprehend and seize nature but also and inseparably in understanding the manner through which Newton's conception apprehends this same nature, but in biased and narrow manners**. The bias is due to the fact that the comprehension of dynamics is clearer with the rapidity concept than with the velocity one (economy of thought: no need of a Lagrangian, accompanied by structural simplicity: the composition law associated with rapidity is additive). The narrow manner corresponds to the fact that Newton's dynamics is reduced to algebraic functions (finite polynomial development) while Leibniz's investigations in conics led him to realise the major importance of what he called "transcendent expressions" whose polynomial development is infinite and that include algebraic functions as particular cases. This fact was much emphasized by Leibniz at various occasions. Philosophically speaking, all those who compare Leibniz's apprehension of the physical world [1-6] with that of Descartes and other contemporaries recognize that the main point is mathematically expressed by the distinction made between "algebraic" and "transcendent" expressions. Let me emphasize that the adoption of the present Leibnizian formulation is not only due to the fact that my comprehension of things is closer to that of Leibniz than Newton, but because of the intrinsic higher understanding provided by Leibniz's methodology. **The comprehension of physics by Leibniz presents intrinsically a number of virtues among which the understanding it provides of the insufficiencies of the Newtonian, Einsteinian and presently available formulations of dynamics**. It is through these insufficiencies that lay the interest of a formulation of a Leibnizian type.

The end of science : a periodical phenomenon!

There exists in the history of physics a phenomenon periodically met and manifesting itself by the positions taken by great scientists starting with Lagrange concerning the achievement of physical science. In the 18th century, Newton is perceived by Lagrange as the "happiest of all mortals" because he found the laws of nature and all what remains to do is "picking up the crumbs of the banquet". Towards the end of the 19th century, Lord Kelvin evokes two small clouds that should be dissipated from the clear sky of physics. The first of these led to Einstein's relativity while the second opened the door to quantum mechanics. In the 20th century, Hawking speaks of the end of physics in a near future and of the knowledge of "God's thought".

Each generation of physicists seems to treat and conceive the world as definitively given, apart from a few details with no importance to which it directs its interest. This conception is certainly advantageous, efficient, reassuring and stable but unfortunately false. Unlike these positions and since the 17th century, Leibniz has foreseen that great men and smart intelligences may envelop themselves with a net where each stitch reinforces the other in such a way that the whole seems marvellously natural; but where hides the first stitch on which all the rest depends? Nobody knows.

*The term "God's thought" evoked by Hawking is repelling to rationality not only because theology is far behind us (even if it was at the basis of philosophy and science) but also because this "concept" is loaded with unnecessary, vague and ill-defined propositions and considerations. The concept of God cannot be avoided in dealing with Leibniz's philosophy but it is sufficiently specified, directly associated with a "principle of action, generation and determination" of entities that cannot be subject to measurement (for different reasons among which their infinite multiplicity). It should be emphasized that **the term "God" has in Leibniz formulation a generative role and it is never introduced to hide our ignorance**. Leibniz uses this term to affirm the existence of certain **transcendence** in physics that cannot be simply grasped empirically through **experiments**. Obviously, on trying to specify the term associated with the* *concept of "God" one is necessarily led to a reduction, but such a reduction should not be confused with the "reductionism of conventional physics". When Leibniz likens God to the ratio of a progression that determines the infinite number of terms of the progression, he surely performs an extreme reduction as to the philosophical idea associated with the concept of "God", but this is the price to be paid to avoid the numerous connotations associated with such a "concept" which is usually extremely vague and underdetermined. Unlike its use in religious creeds and traditions, it should be emphasized that the concept of "God" may have a place in rational thinking only if it is denuded from all superficial constraints attached to it by popular wisdom and pseudo-philosophy. According to Leibniz,* ***the concept of God should be liberated from the stupid connotations associated with it as well as from its consideration as a subterfuge of our happy ignorance.*** *Its true value in natural philosophy lies in its relation to "intelligibility", to a higher comprehension leading to a better understanding of the hidden links that transcend contingent measurable entities at the basis of conventional physical models. It is important to realize that in the same manner as the concept of a "particle" in modern physics differs radically from the initial concept that goes back to antiquity, before being revisited by Newton and his followers, the concept of "God" used as a subterfuge in religious creeds differ radically from the one used by rational metaphysicians such as Leibniz. If Leibniz lived at our epoch he would probably talk of an "emergence principle" or a "generative principle" to account for the ways the different presently available dynamical frameworks emerge as particular cases of the principle of dynamical relativity. However, since we wish to remain as close as possible to Leibniz formulation of things we preferred to use the language used in the 17^{th} century. Moreover, because of the frequent use of the "principle of analogy" by Leibniz one is faced with analogies between theological and physical considerations especially in so far as the problem of substance is concerned, the substance concept being important in physics as well as in theology (trans-substantiation) and philosophy (mind-body problem).*

The recognition by Leibniz of the incapacity of the individual human mind to reach a fundamental truth led him to propose a certain way of research, where the door is open to all rational minds ready to enter into infinity of possible worlds, and a second infinity associated with the points of view on each world. These infinite potentialities are necessary according to Leibniz, even if this does not correspond to a logical necessity but to a contingent: the incapacity of a man's mind to encompass the whole structure of the universe. This proposition is also guided by the idea that if one begins with a too narrow framework and with insufficient reasons to justify the starting point, then one may lead numerous researchers on the wrong track. Here lies the origin of what Leibniz calls the principle of **plenitude** asserting the consideration of all the possibilities and potentialities compatible with a requirement considered to be basic (the principle of relativity in the present work).

A basic difficulty associated with rationality.

This paragraph, devoted to an intrinsic difficulty associated with rationality, is a direct consequence of my rational investigations and the difficulties met by the deadly struggle between two different rationalities and the desire to encompass them into a unified frame. In particular, I found in Ref.[43] some wise suggestions that seemed to me wholly appropriate to the present situation. A man of science is supposed to be rational in the sense that he adheres to the fact that the proof is the politeness of the mind, and that the simple opinion is to be abhorred in favour of sufficiently grounded and justified assertions. However, after some reflection, one discovers that not only rationality is difficult to acquire but its obtainment is often partial because of the finiteness of the human mind and its incapacity to grasp sufficiently global and detailed information at a time. To this, one should add decisive information rarely considered, probably because of its psychological origin although it plays a major role against the development of rationality itself (a sort of internal contradiction). **When rationality is independent of the person that practices it, it seems as an ideal, but whenever it interacts with its bearer it may become a monster.** To seize the problem properly let us make a distinction between a merchant and a professor or a thinker as proposed by P. Riffard in Ref.[43]. When a merchant sells a product, he remains external to it in the sense that even if the product has not a great value or is not of a good quality, the man's value or quality is not affected by this fact. On the contrary, when a thinker proposes a system of ideas, he offers himself to the critics and he puts his entire life in question. He puts on the public market of thought a consequent part of his existence. This is evaluated in dozens of years of desperate unceasing work that will determine the success or the failure of his intellectual enterprise. The stake is so important and the problem is so difficult to master by a simple individual mind (a minute part of a brittle whole) that the rational man, terrorized by the heaviness of the task, terrorizes by assertions surely well-articulated and logically irreproachable but whose **source is suspect** and without a real foundation. Conscious of the fact that the source or the origin are lost in a bottomless abyss, the rational person falls back upon logic releasing it like a dog owner who unleashes his animal. Here lies the dogmatic attitude with its certainty where no hesitation is allowed in fear of detecting the underlying dogmatic creed that says (without proof) more than it knows. Here also lies the origin of the reassuring terms like "it is evident", "it goes without saying" or "it is a fact" so that one does not need to justify anymore what one affirms. If one does not follow the proposed rational scenario, his constructor does not exist anymore since both of them merge into one. If one accepts the existence of the "nought" as a concept, one casts the first philosopher (Parmenides) to nothingness. Parmenides knows that, so he makes metaphysics like a gladiator. Life and death for me and "Being" and "nought" for my philosophy as pointed out in Ref.[43]. After two thousand years of philosophy that Christianity transformed into the servant of theology, humanity entered in the scientific era firstly with Descartes then with Newton, where new foundations of reality are proposed. Parmenides duality between "being" and "nought" is weakened in favour of the Cartesian one through the famous mind-body problem and the Newtonian trilogy: space, time and matter at the basis of our modernity. This manifests itself nowadays by logical positivism. It is true that modern thinking does not adopt anymore the lyrical-logical procedure, allying the flight of oratory to reasoning ensuring to Parmenides a certain seduced audience, but the essence of the question is still the same. One believes that modern persons will not be hoaxed like their ancestors of antiquity and middle ages by a closed discourse formed by a mixture of pleonasms and absurdities. But in reality, the Vienna circle brought an alternative to the lyrical-logical procedure of the ancients. This physical-linguistic alternative uses the same mode of functioning to penetrate the spirits. Terrorized by metaphysics, the Vienna circle will terrorize by the syntactical logic disguised in physicists and wearing the dress of "metaphilosophy". The Vienna circle is taught in all the histories of philosophy and of science. One encounters disciples, dissidents and rivals, while one knows that most of the ideas are uncontrolled when they are not intrinsically false. As to the opposite circle headed by H. Reichenbach who actively defended Leibniz's epistemology and philosophy of nature, is much less known and recognized. Its main error is that it does not manifest itself enough loudly. In philosophy and science, like in the market place, one listens to the one who raises his voice more than the others.

During the pre-scientific era, rational theology used to guarantee the truth of the moment. Nowadays there is more diversity through the recourse to the physical trinity: (i) experimental physics (ii) formal physics (iii) axiomatic physics. The first aims at leading dumb: experience leaves one without voice. Experience affirms it, hold your tongue. The second aims at leading blind: do not look at nature but think at my equations. As to the third and last one, it aims at rendering deaf: register my basic commandments and do not listen to anyone else but to what is deducible from the first axioms.

After all this criticism, one should propose a solution as to the exact location of truth which does not seem to be localized in the realm of empiricism, or in that of formalism or still in the one associated with first axiomatic necessary considerations. According to Leibniz, the truth is more widespread than what is usually believed. The very idea of scientific truth is to be looked for at different locations at the same time, according to the scale one is looking at. Here is the main lesson that one may draw from Leibniz's epistemology. This multiplicity of scales and points of view leads to a substantial difference, as compared to the usual discourse on physical reality directly attached to one modality of measurement or another. With Leibniz, the number of the different modalities of existence are infinite, and this infinity is essential because it allows to discover an order that can only be placed in evidence if one takes seriously the infinite multiplicity of points of view. Without this fact, no recurrent series can be exhibited, infinity constituting the essence of the indefinitely repetitive procedure at the basis of any kind of recurrent series. Apart from this quantitative infinite realm, the adoption of an undetermined infinity through qualitative arguments allows to obtain a smooth passage from quality to quantity through a semi-qualitative or equivalently a semi-quantitative realm, absent from all previously given formulations on dynamics. This fact is a direct consequence of the distinction between the two realms of essence and modalities of existence, where the latter do not need to be given quantitatively to obtain a quantitative expression of the essence of dynamics.

Thus, the degrees of freedom proposed by Leibniz and developed all along this work show that unlike what is usually believed, Leibniz's approach is not purely metaphysical but it constitutes an antidote to dogmatism – which imposes itself without a proof and says more than what it knows – leading to new explanations and novel explorations.

Essence, modalities of existence and their Harmony: a musical analogy.

The main thesis defended in this work is that as long as measurement is imposed on motion and dynamics (at the basis of physical science) the door remains closed to a possible higher intelligibility than that provided by abusively called **theories** while they are only **models**. The distance between the abstract terms associated with a "model" and a "theory" may be understood analogically by the distance between the concrete situations associated with a "monophony" (one musical instrument) and "polyphony" (an orchestra: symphony). One musical instrument may be sufficient to inform the auditor about the name of the "symphony" through the partition one is playing. But, as long as one hears the instruments one after the other, the unity of the symphony is lost and the pleasure it induces is absent. The only way to appreciate a symphony is to hear the instruments simultaneously since, only in this case, one is able to discover the richness produced by the different instruments echoing each other harmonically. Here lies the real meaning of harmony in Leibniz's methodology. It is not in the sound of the instruments heard separately but in the correlations between sounds and silences produced by the multiplicity of instruments. In the same manner as all the instruments are not of equal importance (some are more basic than others), the different models are not of equal importance. In the same manner as there is not necessarily only one basic instrument there is not necessarily only one basic model. It is important to realize that, in physics, we usually play monodies rather than symphonies. The monody associated with the "usual rationality" has been considered to be the unique one at the basis of dynamics until recently. Numerous historical, epistemological, conceptual, mathematical and physical arguments tend to meet, showing that the existence of another possible rationality (called, here, the "emergent rationality") is not an illusion as believed for a long time, but a reality as shown recently by many theoretical, mathematical and experimental physicists. In addition to these two rationalities (monodies) one may distinguish a third one (provided by Taylor and Wheeler and based on celerity instead of velocity or rapidity as shown in this work). Behind these different monodies Leibniz's methodology reveals the existence of a symphony (polyphony) that orders the different elements showing the place of each one with respect to the other and revealing the conductor of the orchestra. In Leibnizian dynamics, the conductor of the orchestra corresponds to the ratio of the geometric progression that indicates the place of each point of view and its articulation to others. In the same manner as the conductor of the orchestra does not belong to the elements of the orchestra, the ratio of the geometrical progression does not belong to the terms of the progression. It transcends the different points of view constituting their essence and the reason of their existence without taking part to the contingent and measurable points of view. This example is the one given by Leibniz himself as indicated by C. Frémont to distinguish between "contingency" and "transcendence" possibly associated with different contexts including physics. (In physics, Leibniz was also interested in magnetism: a magnet orders the initially disordered minute metallic pieces and plays the role of a "principle of action" that transcends each metallic peace ordering them in a harmonious way.)

If these considerations that justify analogically the net distinction between "essence" and "modalities of existence" remain highly unknown by physicists, it is because physics can do without it. The main interest (appearing first) in the Leibnizian methodology is "intelligibility" more than "efficiency", "explanation" more than "exploration". But, ultimately, both attitudes go hand in hand and contribute to a deeper understanding as well as to a more extended action.

Measurements, norms and normality.

The question of measurement is not specific to the scientific world. All kinds of measurements give rise to norms, at the basis of any human society where people are compared to each other and things evaluated and exchanged. It is quite impossible to make abstraction of measurements and norms. When the norm is the statistical reflect of a huge number of data it becomes what one calls "normality". Most individuals are those whose attitude does not vary significantly from normality by definition. In order to place each individual at the "right place", many scientists began to measure the human body, the form of the eyes, nose and ears as well as the circumference of the head etc. These measurements were aimed among other things to detect criminals before committing their crimes. One may refer to Bertillon's measurements of skulls (headquarters of Paris police) in order to distinguish the criminal identities from the others. This belief in an **innate tendency** to criminality discoverable by such simple, naïve measurements considering that the **outside reflects the inside** was more than a deviation, it was a monstrosity. The same holds for the measurement of the intelligence. The mental age deduced from Binet's test served as a model to modern intelligence tests. This fixed manner of looking at things at one moment independently of the psychological state of the individual, of its origin and evolution, shows the limits of such methods of investigations.

This holds for scientists in their relation to the "scientific truth" of the moment. It should however be emphasized that, contrary to the recognition of different norms associated with different cultures and civilisations living at the same epoch, one recognizes only one norm associated with the scientific enterprise whatever the culture from which a scientist is issued. That is what is meant by "scientific universality" which is the same all over the world. Obviously, what is believed to be true today may be wrong tomorrow, but there are a number of truths that are not invalidated but only better specified. The following form of Pythagoras theorem $a^2 + b^2 = c^2$ remains valid only in a Euclidean framework while it takes other forms in non Euclidean geometries.

Possible extension of the principle of analogy to link "physics" to "metaphysics".

Most of physicists believe that the different affirmations and articulations (discovered through history on a subject matter) have to be integrated by the individual who pretends to bring a new light on the considered subject but this fact is not always true. If it had to be absolutely verified, philosophy could not be of any use to physics: philosophers do not know physics in detail but only through its basic concepts. There is at least one principle that shows that one may bring fruitful ideas to one discipline even if the discipline in question is not completely mastered and known in detail. This principle is none other than the **"principle of analogy"** which is at the basis of human thinking and largely used in physics. It is curious that the physicists who know that very well (they use it to pass from mechanics to electricity and magnetism, from translations to rotations and torsions as well as in many other areas of physics) refuse its applicability beyond physical situations. As long as one remains in the field of the scientific world, the **"principle of analogy"** may be used fruitfully but no analogy would be allowed between metaphysics and physics. Kant's distinction between what deserves to be studied by science and what should be rejected in the realm of metaphysics plays a major role in modern science. There is no scientific program whose goal is to benefit from the thousands of years of thought on "transcendence" and "immanence" (produced by metaphysicians and philosophers as well as ancient physicists and mathematicians). On the contrary, in order to depreciate some theoretical works, these are considered to be metaphysical (in a pejorative sense). This broken link between the "metaphysical" and the "physical" realms deserve to be re-examined seriously for what is metaphysical today (lack of a demonstration of its "physicality") is not necessarily unphysical forever.

Necessity of dogmatic creeds to ensure stability.

It should be emphasized that science criticizes the dogmatic creeds of religion (punishment its opponents) but at the same time it reproduces exactly the same scheme to ensure its stability and survival. It is clear that if science would accept anyone that proposes extraordinary ideas with the unique aim of producing new imaginary or illusory things, included in the scientific literature, then science would collapse. This explains the strictness of its evaluation concerning the "abnormal" individuals. One of the main difficulty at the individual level, is that the time elapsed may be very long between the discovery of a potentially fruitful idea (too distant from "normal" thinking), and the proof of its rational physical validity (necessitating a direct link with the basic physical principles). **The discovery is one thing and the proof of its truthfulness** (through a mathematical demonstration) **is another one**. This explains that most of the times**, the origin of many fruitful ideas evaporates like water infiltrating hot sand.** The origin remains unknown unless the problem is well-identified, and some people are already working on the subject matter where a question needs to be answered. In such a case, one may easily find the origin of the correct solution. Most of the basic questions posed by philosophy have never been answered, and some of them were finally considered as unanswerable or ill-posed. If one works on a question considered as being unanswerable or unknowable – and hence metaphysical – then such a person will not be listened to. The ways out of this situation are, either to work on well-identified questions or be patient, transcending contingency and tracing one's own path with a look from time to time at the evolution of the thoughts in the contingent world, in order to keep a link with their progression.

What is a vice at some epoch may turn to a virtue at another one.

It was a great surprise to me to find out that the logic behind Leibniz's dynamics was rejected by physicists and epistemologists in the realm of metaphysics that deals with the unknowable or the unanswerable, in spite of the fruitfulness of this same logic in mathematics and particularly integro-differential equations. How is it possible to a mind to be rational in mathematics and irrational in dynamics, expressed through this mathematics? Leibniz spent his life emphasizing that his assertions and his critics of the dynamical framework provided by his contemporaries are based on his discoveries of some mathematical properties unknown by the others. Nowadays, in reason of the "unreasonable efficiency" of mathematics in the discovery of new laws of physics such a situation could not be reproduced. No one would accept the rejection of the works of great mathematical physicists who develop symplectic and non-commutative geometries because what they produce go beyond the immediate understanding of most physicists. Yet, that is precisely what happened to Leibniz. What was considered to be a vice in the 17th and 18th centuries, turned to a virtue in the 20th and 21st centuries. Some may be tempted to retort that many mathematicians do not care for physical, biological or any other application; Leibniz might be one of them: but this is historically false. Leibniz spent his life developing a conceptual framework apt to apply to different contexts (universal characteristics). Moreover, in a letter addressed to Huygens, who was his teacher in dynamics, Leibniz presents the finality of his mathematical discoveries as a bridge for a better understanding of natural phenomena. In addition, one should keep in mind that Leibniz was the last scholar that was able to deal with different physical and non-physical contexts in a unified way. Such a behaviour was not possible later on, not only because of the accumulation of knowledge and the impossibility of a human mind to encompass all the information as usually asserted, but because, unlike the Aristotelian paradigm, the Kantian one broke any possible link between metaphysics and physics. These contexts have been separated by an insuperable abyss.

The specificity of Leibniz : Principle of relativity and principle of plenitude.

The main difficulties of this work are conceptual, and go against popular scientific wisdom, based on the fact that, during thousands of years of philosophical thinking, none of the philosophies showed its relevance in dealing with natural phenomena and particularly with one of the oldest questions relative to motion. However, this ascertained fact does not mean that philosophy is not apt to cope with the problem of motion. It only shows that all the philosophers except one (Leibniz: philosopher, mathematician and physicist) were not prepared enough to deal with this rather difficult question and that the one able to encompass simultaneously the **relativity principle** and the **principle of plenitude** was not recognized until very recently. **Among all the philosophers and scientists only Leibniz defended with an equal importance the two facets of the problem of motion given by the principles of relativity and plenitude.** All the given answers, since Aristotle turned out to be either intrinsically false (as those presented before the epoch of Galileo) or insufficiently articulated as what happened during the Newtonian period. Since Einstein's rediscovery of the principle of relativity, this principle was discussed more seriously in the 20th century, but the principle of plenitude associated with it is until today practically unknown. The recognized physicists interested in the problem of motion like Galileo, Huygens and Newton in the17th century, Lorentz, Poincaré and Einstein in the 20th century did not suspect the existence of a **scale recurrent law** (advocated by Leibniz) that constitutes a physical justification of the (initially metaphysical) **principle of plenitude**. As shown in this work, it is the discovery of such a scale recurrent law which is at the basis of the **discovery of an ontological order, hidden behind the epistemological disorder** encountered through the various interpretations and developments of various models unarticulated to each other, in order to find a true theory of motion as proposed by Leibniz. This scale recurrent law reveals the different presently available points of view on motion by repeating itself indefinitely and unfolding thus one point of view at each step. The different sequences are numbered in such a way that, to each number, is associated one specific point of view corresponding to one dynamical model. Such a numbering is absent from the usual modelling and looses its "raison d'être" since in dealing with one model or another there is nothing to number. Its existence is intimately related to the global correlation between the different models. It is not inherent to anyone of the specific models or to any of the internal correlations between its elements but it transcends all of them.

The "Leibnizian transcendence" opposed to the "metaphysical transcendence" as well as to the Kantian "transcendental object".

The Leibnizian form of transcendence is to be distinguished from the strictly theological one whose aim is to reach the "Absolute" and the "Infinite". For Leibniz, the "Infinite" is not unreachable but it may be associated with contingency. Philosophy does not consist in escaping from contingent reality at the search of another world. It consists more simply to accede to "ultimate causes" behind a contingent reality. If the idea is simple, the task to reach such and ideal may be very difficult to put into practice. According to Leibniz, a **true philosophy** does not reveal a new world replacing the contingent one, but a **new significance of the world** in which a place is reserved to **liberty** or degrees of freedom, **compatible with the basic necessities** (in dynamics, the principle of relativity and conservation properties). Leibniz was against the philosophers whose reflection was situated at a religious transcendent level despising all what amounts to action. He aimed at getting beyond the opposition between reflection and action, so that any reflection becomes an action and any action requires a reflection. The access to “truth” is conditioned by our position in the world that imposes some absolute necessities but keeps the door open to a multiplicity of degrees of freedom (plenitude principle). According to Leibniz, one should not confuse what amounts to **“reality” itself** and what amounts to the **different projections produced by the “minds” on this reality**. Leibniz’s dream was to find out a way allowing him to reach the core of dynamics without passing necessarily by one projection or another as done empirically by Newton, rationally and analytically by Lagrange and conceptually or philosophically by Kant. The different methods proposed, [not only by space-time physics (Newton, Lagrange, Lorentz, Poincaré, Einstein, Minkowski, Taylor and Wheeler, etc.) but also by the recent dynamical frameworks (dealing mainly with group theory: Comte, Lévy-Leblond, Bacry, Provost etc.)] follow the conceptual framework traced by Kant, according to which one needs to account for some external remarkable property to account for dynamics. Such a remarkable property does not belong to any dynamical concept. It is a sort of extra-concept (or non-concept) called by Kant an “a priori form of intuition”, distinguished from the conceptual realm associated with Kantian categories. It is also what Kant calls a “transcendental object” needed to do physics although it cannot be deduced from any physical experiment. With the advent of Einstein’s relativity many assertions provided by Kant as to the structure of space and time (inherited from Newton’s mechanics) turned out to be false, but the general structure remains valid. If one replaces the Newtonian space and time by Einsteinian space-time, one continues to satisfy the general Kantian arguments provided one adapts them to the new framework. Let us note that what counts is the notion of velocity (a ratio of a length over a duration) which continues to be fundamental in Newtonian and Einsteinian physics. In modern dynamical formulations, the velocity dealt with through the Lagrange-Hamilton formalism is replaced by the rapidity embedded naturally in group theory. In spite of this change of perspective where the space-time motion (based on a ratio) is replaced by another point of view (additive composition law), this new framework remains, nevertheless, compatible with Kant’s transcendental object. One simply replaces one remarkable property by another one. Dealing with dynamics requires, for Kant, the selection of a specific cornerstone on which the rest is built rationally using an appropriate methodology. Here lies the difference between the “Leibnizian transcendence” and the “Kantian transcendental”. In this last, case one needs in a way or another to recourse to some quantitative object on which dynamics should be built. For Leibniz, this is not necessary; one may deduce the structure of dynamics without adopting any quantitative point of view attached to one remarkable property or another. Leibniz’s transcendence affirms the possibility to transcend any specific point of view or any remarkable property. Thus, the Leibnizian transcendence differs radically from the Kantian one (transcendental object): it rejects the recourse to any remarkable property or point of view on which dynamics is erected. (These remarkable properties occur in physics through so-called “simplicity principle” or “simplicity criterion”. The use of rapidity for instance is based on the additive composition of motion which is the simplest composition law). Unlike Kant’s conceptualization (followed by all available past and present dynamical models), Leibniz’s methodology aims at **transcending the “simple” and the “complex”** considered to be associated with a subjective criterion that the mind projects on reality. To deal with reality itself independently of any specific projection, one should accede to a quantitative form relating the conserved entities without any specific quantitative account for motion (through velocity, rapidity, celerity … non-conserved elements). Instead of relying on a remarkable property or another, Leibniz's goal is to generate different remarkable properties, each of which would constitute one point of view on motion. To do this, one has to **reverse the Kantian finite methodology replacing it by a potentially infinite one.** Infinity is essential for the Leibnizian methodology for its absence eliminates the possibility of constructing a generator of remarkable properties by use of an iterative procedure, infinite by its very nature. If one digs deeper in the main Leibnizian and Kantian philosophies, one discovers that unlike Leibniz, Kant does not believe in the "univocity of being" according to which the "human understanding" differs from "God's understanding" only by degree and not by nature. This explains Leibniz's optimism as to the possibility for a human intellect to reach the ultimate causes hidden behind a scientific discipline such as dynamics. If Kant does not reject the existence of hidden causes as such, he nevertheless rejects the possibility of reaching them rationally. Such hidden causes are thought of as purely metaphysical and out of reach by any human understanding: no rational procedure is available to unearth them leading thus to the root of every matter. In other words, from a scientific rational standpoint, there is nothing to look for behind the curtain. As long as one perpetuates the use of the presently available analytical models, one finds that Kant is right. However, this does not prove the non-existence of a higher rationality. This only proves that the mathematical methods accounted for are too poor and narrow for such investigations. This also proves that the present scientific platonic methodology puts the cart before the horse, in favouring mathematical methods on conceptual thinking. Modern dynamics, based on group theory, starts its investigations by noting that the use of group theory allows (thanks to a certain theorem discussed by Lévy-Leblond and Comte) to replace the rather complicated composition of motion by a simpler one which is additive. This led to the birth of rapidity on a rational ground. However, as long as predetermined mathematical fields are used to deal with a subject matter, one remains imprisoned in the structure of this mathematics (velocity for Lagrange-Hamilton formalism, rapidity for group theory and celerity for metrical geometry with invariants). The only way out of this situation (where each mathematical methodology favours one point of view considered to be natural), is to stick to the dynamical framework considering the absolute necessities without which dynamics collapses. Dealing with the realm of necessity, this dynamical framework reveals the existence of an infinite number of degrees of freedom compatible with such a necessity. Instead of imposing a specific mathematical methodology leading to select one appropriate parameter to deal with motion, one constructs a yet non-existent mathematical methodology able to encompass the above-mentioned infinite multiplicity of degrees of freedom. Here, the act of birth of physics and its appropriate mathematics are simultaneous.

Let us recall that, in the study of mechanics in the 17^{th} century, the notion of a derivative was suggested by the study of the instantaneous velocity: the physical framework was developed simultaneously with the mathematical one. This explains the superiority of Leibniz investigations as compared to his contemporaries. He was aware of the importance of both the conceptual framework and the formal one. The formal framework imposes itself not only to communicate the necessities, intuition and concepts that one has concerning a subject matter, but also to advance in ones investigations. It is only when the concepts are put in a formal manner that one discovers the remarkable properties impossible to reach without such an explicit development (that should be logically irreproachable and devoid of any contradiction).

The possibility of reaching the core of dynamics without any use of a predetermined parameter associated with any **a priori** form of intuition shows the limit of validity of the Kantian paradigm (and with it the usual analytical methods of physics). Leibniz's belief in the unity between the human and the divine intellect (a heretical position for most religious authorities) led him to look for methods capable to grasp dynamics with a "God's eye" view, transcending any specific projection or simple vision such as those provided by the presently available rationalities. Because of such heretical and unphysical (not subject to direct measurement) ideas, the price paid was very high: Leibniz ended his life in a great solitude, rejected from both the physical and the theological communities.

BASIC HISTORICAL, CONCEPTUAL AND MATHEMATICAL POINTS.

In order to shed some light on the general structure of this formulation, we shall summarize some basic historical, conceptual and mathematical points intimately related to dynamics.
Noting the main difference between what is usually asserted concerning Descartes dynamics and its present interpretation in the light of the Leibnizian methodology, it seems to us important to specify who the Descartes of this work is. Then, a correlation is made between conceptual aspects and mathematical ones in the discovery of some basic elements of physical reality.

Who is the Descartes of this work?

It is of the greatest importance not to confuse between the different facets of Descartes thought. Descartes as a philosopher differs from Descartes as a scientist, since many Cartesian general ideas turned out to be fruitful, while their application by Descartes himself and the conclusions deduced from such applications were deficient. This is not only true in dynamics based on the conservation laws but also in different other disciplines. In addition to these two facets of Descartes, one should distinguish between two other facets associated this time with the manner one interprets Descartes dynamics. The physicist does not interpret Descartes in the same way as the historian of science, so that a number of confusion results from this fact. In physical studies one presents Newton, d'Alembert and Lagrange as being the ones who made a fruitful synthesis of Descartes and Huygens dynamics by taking the concept of "quantity of motion" also called impulse from Descartes and the "living force" which corresponds to the double of kinetic energy from Huygens. This assertion constitutes a great reduction of historical reality, since what Descartes calls "quantity of motion" does not coincide with impulse but with its absolute value which is something totally different, from a mathematical as well as from a conceptual standpoint. In one case one deals with an odd function while in the other, one deals with an even function (generalized irregular function). Conceptually, only even functions are apt to conform to the notion of "active substance" that corresponds to a positive value whatever the direction of motion. In addition to this discrepancy between what happened effectively and what is believed by most physicists, the function associated with Descartes dynamics is irregular while the one associated with Huygens dynamics is regular. This fact is of a great importance for in the absence of this consideration one cannot understand Leibniz's critics of Descartes incapacity of articulating the state of rest to that of motion continuously. Leibniz's critics seem totally unjustified if one is not conscious of the fact that the Cartesian dynamics corresponds to an irregular function. This irregularity is so important that it leads to a serious criticism of the usual physical methodology. The logic behind this criticism runs as follows: an irregular function belongs to what moderns call "generalized functions". Now, all the methods developed by conventional dynamics are confined to regular functions and hence to a too narrow framework unable to encompass Cartesian dynamics. Thus, if the adopted methodologies are able to judge Newtonian, Huygensian, or Einsteinian dynamics because of their regularity, these are not apt to judge the Cartesian one. Yet, Descartes dynamics is rejected, but as just shown, its rejection is not possible by use of the presently available methods used for the evaluation of dynamics. Here lies one of the main discrepancies and fallacies of the "usual" and "emergent" rationalities that affirm more than what they are allowed to judge. Here lays their dogmatic character asserting what they can **neither affirm nor deny**. This impossibility of a rational judgement requires the construction of a wider and appropriate framework capable to deal with Descartes dynamics on a rational ground. With such a construction not only many Leibnizian positive critics become rationally justified but the solutions proposed by Leibniz turn out to be fruitful. Their fruitfulness is not restricted to a better understanding of the past of dynamics, but also to a deeper comprehension of its present state and its possible extensions for future investigations.

A fruitful marriage between "conceptual" and "mathematical" considerations for a better understanding of physics.

In this Section, it is shown that the "mathematical" and the "conceptual" features are closely correlated to each other. In the first example devoted to the inquiry about the possibility of a new rationality, the mathematical is predominant. It suggests the existence of such a rationality that remains to be confirmed from a conceptual standpoint since no rationality is conceivable without a firm foundation. Such a foundational ground lies (by its very nature) beyond the mathematical equations, which only ensure the coherence of the new rationality and constitute its manifestation for the resolution of the physical and practical problems. In the second example devoted to the inquiry about the possibility of a multiple rationality transcending the presently available ones and encompassing them into a wider whole that explains their relations, the "conceptual" is predominant. The "mathematical" comes at a later stage as a necessity. It imposes itself because physics is not conceivable without mathematics. The third example concerns the important role played by differential equations in the establishment of the principle of dynamical relativity.

In the following three paragraphs we shall not enter into details given elsewhere in this work: we shall simply focus the attention on the basic ideas and their relation to the "mathematical" and "conceptual" features.

What lies behind the belief in a possible new rationality?

In dealing with Newtonian parabolic dynamics one discovers the following relations:

$$E = 1/2mw^2 + \text{Const.}\quad p = dE/dw = mw$$

here impulse p derives from energy. The same property holds in dealing with Einsteinian hyperboplic dynamics since (introducing rapidity) one may associate the hyperbolic structure with

$$E^2 - c^2p^2 = \text{Const.}\quad , p = E' = dE/dw$$

from which one may deduce

$E = mc^2 \cosh(w/c) \quad p = dE/dw = mc \sinh(w/c)$

These remarkable mathematical properties where impulse derives from energy in the parabolic as well in the hyperbolic case, satisfy the well-known qualitative relation between Newtonian and Einsteinian dynamics since for $w/c \ll 1$, Newtonian dynamics may be regarded as a local solution of Einstein's one as is well-known. However, in spite of this order, one should emphasize the fact that **the above relations contradict the space-time rationality where impulse does not derive from energy** but from a Lagrangian. Thus, the parameter w cannot be associated with the velocity concept that verifies $p = dL/dv$ and $v = dE/dp = dr/dt$. The above well-known mathematical properties, which amount physically to a direct link between impulse and energy through the parameter w, can only be retained if one constructs another rationality than the one provided by Lagrange and Hamilton. The price is very high because of the extraordinary role played by the Lagrange-Hamilton formalism especially in the development of modern physics (particularly after the advent of relativity theory and quantum mechanics). Thanks to the works of J.M. Lévy-Leblond and C. Comte, I realised that the relations I had established by use of remarkable mathematical properties, had their physical counterpart in a new rational framework that brings a new point of view on motion. This rationality is the one that we called "emergent rationality" distinguishing it thus, from the "usual rationality" of Lagrange and Hamilton. **The existence of two different points of view on the same dynamics suggests looking for some procedure that allows reaching the core of dynamics without necessarily passing by one point of view or another**. It also suggests looking for other remarkable points of view and possibly to link all of them by some higher methodology capable to embrace all of them through one compact formula. This last conceptual idea will acquire a mathematical justification in the next paragraph.

Here, the mathematical precedes the conceptual, for one firstly sheds light on the mathematical properties before looking for a possible conceptual framework, physically defendable and compatible with concrete physical measurements. In the next paragraph, we shall show that contrary to what occurs here as to the primacy of mathematics, the priority will be given to the conceptual standpoint where mathematics comes at a later stage to confirm the basic intuition by constructing it effectively.

What lies behind the belief in a possible existence of a multiple rationality including the "usual" and the "emergent" ones as well as other perspectives?

One way that materializes Leibniz's intuition – asserting that what is considered to be "one" is effectively "many" – consists in showing that the extension of linearity **is** not **nonlinearity** as usually done, but **nonlinearities** as shown in Appendix H. An infinite number of nonlinear curves may converge towards a fixed point following a unique tangent, so that in the vicinity of the fixed point the infinite curves become fused together, getting the impression that there is no multiplicity at all. In spite of the apparent mathematical property at the basis of this reasoning, the germ out of which such an idea appeared lies in the conceptual distinction between "substance" and its "modalities of existence". Without such a conceptual distinction, the mathematical property remains useless. In order to better grasp the priority of the conceptual standpoint let us note that the first criticism of such a mathematical idea – as to its relevance to physics – is that

the passage from a unique solution to a multiplicity of solutions leads automatically to a lack of determination and hence to an impossibility of prediction. One is thus cast into the realm of pure mathematics, far from any physical consideration. This reasoning is the one that the 17^{th} century physicists addressed to Leibniz's multiplicity where the replacement of a unique structure by a multiple one is not conform to the physical predictive requirement. The only way out of this difficulty, requires associating the **multiplicity** with **points of view** on a given reality and not with the reality itself. This leads automatically to **the distinction between a multiplicity of "modes of existence" and a unique "substance"** obtaining thus a rational framework through which a **multiplicity does not contradict predictability anymore**. But this is possible only under certain conditions where one is committed to make additional distinctions absent from the usual physical methodology. With this new conceptual framework, one goes beyond the usual understanding of physical formalisms attached from the start to the development of a unique point of view. As shown in this work, it is possible to construct a theory capable to encompass different models among which those associated with the "usual" and "emergent" rationalities.

What lies behind the differential equations associated with dynamics.

For brevity and in order to take into account the most known dynamical approaches, we shall confine ourselves to the particular form associated with the dynamical relativity principle, where the parity requirement usually associated with space isotropy is imposed. This restriction does not interfere negatively with the general discussion concerning the important role played by the differential equations at the basis of this formulation. It only simplifies the discussion and the comparison with the main presently available formulations of dynamics.

A third order differential equation.

One of the main Leibnizian ideas about unity is that a unique differential equation possesses different kinds of regular and irregular solutions depending on the limit conditions imposed on it. This is exemplified here through the comparison of the different dynamics starting with Descartes one (17^{th} century) and ending with very recent ones (21^{st} century). If one considers the following third order equation: $pp''' + 3pp'' = 0$, then one may show that this equation is satisfied by a multiplicity of frameworks, five of them being well identified dynamical formulations. Two of them have been developed these last years and are associated with "doubly special relativity [7-8] and two others go back to the 17^{th} century: Descartes and Huygens (or Newton) dynamics. The fifth one that corresponds to Einstein's dynamics occupies an intermediate position chronologically and structurally. It is remarkable to note that the above mentioned third order differential equation could have been derived since the 17^{th} century (by conciliating Huygens and Descartes dynamics) if the Leibnizian conciliatory attitude had been adopted. Once this equation is revealed one is then led to exploit its structural potentialities discovering thus different other solutions.

It should be emphasized that the conciliatory Leibnizian attitude was not an irrational or superficial one. Its rationality was particularly based on his knowledge of conics. Leibniz asserted at different occasions that the cuts one may obtain from conics reveal how two different local worlds, irreducible to one another, may nevertheless be included in a unique entity. This affirmation applies marvellously to Descartes and Huygens dynamics both of them corresponding to particular cuts of cones. This is one of the main reasons that render Leibniz epistemology useful to modernity.

Let us recall that in the above paragraphs two opposite situations have been revealed: In the first one, the "mathematical" precedes the "conceptual" while in the second the "conceptual" precedes the "mathematical". Here, the two go hand in hand, since there is no reason to look for a unique mathematical differential form if one is not at the research of some conceptual unity hidden behind the diversity of solutions, each one being a particular cut in a more extended whole. It should be emphasized that the discovery of a certain unity is not sufficient to get a better understanding of the underlying physics. Such a unity is purely structural, and only when the physical principle behind such a unity is identified that one gains in physical intelligibility. Modern mathematically oriented physics has accustomed us to content one self with purely structural efficiency, but the physicist wishes to get a better grasp of what is going on **behind the efficiency of mathematical structures**. Here, it is the **principle of dynamical relativity** initiated by Huygens (formalized and conceptually extended by Leibniz) that lies behind the above mentioned third order differential equation. It is remarkable to note that, Leibniz had at hand both, **the structural and the conceptual features**. He inherited the first from Pascal's work on conics and the second from Huygens's work on dynamics through his positive use of the principle of relativity. It is worth noting that after having put the emphasis on this principle, it was forgotten for centuries and rediscovered by Einstein in the beginning of the 20th century as noted by some physicists and epistemologists (particularly J.M. Lévy-Leblond in his pedagogical work of vulgarization of basic physics).

<u>*A second order differential system of equations.*</u>

The principle of dynamical relativity is expressed in a compact manner if it is cast into a system of second order differential equations including an infinite number of solutions, each one related to one point of view of motion. Let us recall that this writing in the form of a second order differential system is intimately related to the requirement of two and only two conservation laws as shown at length in this work. More precisely, it is the constraint that one has to impose on the second derivative that leads to a second order differential system. This system, composed of an infinite number of equations, is linked to the above-mentioned third order unique differential equation in reason of a mechanism of compensation that allows one to pass from an extrinsic subjective formulation – relating the conserved entities, energy and impulse, through different possible parameters each constituting one point of view on motion – to an intrinsic objective formulation where the different points of view are eliminated in favour of a direct relation between the conserved entities. This passage from the "infinite many" characterizing subjectivity (Leibniz) [or an "outside vision" (Bergson) as shown in Appendix Z], to the " only one" characterizing objectivity through trans-subjectivity (Leibniz) [or an "inside view" (Bergson)], correspond to the passage from the "subjective version" of the principle of dynamical relativity to the "trans-subjective version" developed at length in the first part of this work.

This paragraph clearly shows the importance of dealing with differential equations in dynamics. They play a major role for a better understanding of this science, usually dealt with according to the unique point of view associated with the velocity concept. This point of view, in its usual development, hides the important role played by differentiation which is inherent to the principle of relativity. When dealt with at the most fundamental level, this principle operates through the notion of a "generalized derivative" which turns out to be a "generator of conservation laws" apt to be regarded under different perspectives. It is the discovery of a possible unity behind the different available methodologies and perspectives on motion that led us to emphasize the importance of differential equations in dynamics. This plays a major role in both unities: the one associated with points of view and the one related to possible worlds. In dealing with the multiplicity of inclusive points of view, one uses the system of second order differential equations. In dealing with the multiplicity of exclusive possible worlds compatible with the dynamical relativity principle, one uses the unique third order differential equation that leads to different solutions, each one corresponding to one possible world. Among the possible worlds one gets various possibilities. Some worlds are regular while others are irregular. Some are local other are semi-local or semi-global while the richest ones are global. The passage from global to local entities is performed by letting one of the constants revealed by the process of integration vanish or tend to infinity. Let us note that a constant revealed by the process of integration is not necessarily an integration constant. It may be a combination of different integration constants since an n^{th}-order differential equation provides n different integration constants. A physical entity may result from the combination of different constants.

Same facts but different words. Same words but different facts.

Same fact but different words.

According to whether one adopts the "usual rationality" or the "emergent one", the same fact turns out to be dealt with through different words. This leads to a certain epistemological disorder if one does not pay a special attention at the used terminology. In particular, the term "motion" which was unambiguous in Newtonian dynamics becomes ambiguous in Einstein's one, unless one specifies what is meant by motion. In particular, it is not sufficient to specify that what is meant by motion is the ratio between a length and duration. This specification eliminates the rapidity parameter which does not correspond to such a ratio. Worse, the situation remains nevertheless ambiguous as long as one does not specify the kind of duration at work. Is it the one associated with the clock attached to the train in motion or is it the clock associated to each railway station (attached to the earth). Obviously, this makes no difference in Newton's dynamics because of the absolute character of time associated with this dynamics. Even the rapidity concept which is not definable through a ratio becomes compatible with such a ratio, because of the degeneracy of Newton's dynamics where the different parameters become fused together. This fact was at the basis of a number of controversies that led to an epistemological disorder in the history of science, and that the present work contributes to lessen through the idea of multiplicity of points of view on a given reality which is rooted deeply in the present formulation, constituting one of the cornerstones of the present approach. It is treated from the inside through a specific formalization and order, and not from the outside as in presently available dynamical models. This distinction measures the distance between Leibniz's epistemology and the usual one.

Same words but different facts.

The inverse assertion is also responsible for some epistemological disorder, the same words being sometimes attached to different facts. More precisely, when dealing with motion, the Newtonians and the Leibnizians often used the same words but the meaning was totally different from one argument to the other. In particular, the expression "divine transcendence" (which was not a taboo in the 17th century) that corresponds to what is beyond any human experimental reach, was used in connection with motion in two completely antagonistic ways. Let us recall that the discussion of motion in space was associated with theological considerations as shown by various historical studies (among which Max Jammer's book on the concept of space [23] starting from antiquity until recent times including the Jewish, Christian and Muslim traditions, in addition to scientific considerations). For Newtonians, the fact that the velocity was possibly infinite did not constitute a real problem, because what is unreachable by a contingent being is possible by a transcendent one. For the followers of Leibniz, the simple fact that a body may occupy simultaneously two arbitrary locations in space constitutes a negation of time, for time is introduced in order to avoid such a miraculous possibility. Geometrically speaking, a body is located at one place and its presence at another place is possible provided a certain amount of time has elapsed. Thus, one essential property of time is that it allows to a body to occupy two different locations without loosing its identity. The presence of a body at two different locations at the same time was considered, by Leibniz, as a miracle and miracles had to be rejected from the realm of rationality. The Leibnizians could not accept this form of "transcendence" associated with space and time while the Newtonians pretended that what is unreachable by humanity is reachable by divinity. Leibniz would accept this last proposition without accepting the irrational way by which it was applied in space-time physics. In order to realize in what sense the above sentence can become rational, one should look at things quite differently, placing oneself beyond the definition of motion through space and time, considering this definition as a point of view among others.

"Transcendence" in Leibniz's conception acquires a totally different meaning. It is intimately related to the incapacity of any human to measure an infinite number of points of view on a given reality. The human nature is finite in so far as its will and power are concerned but it may reach infinity and hence divinity as to its intellect. In brief, according to Leibniz, the divine and human characters do not differ by nature but only by degree with respect to intellect. Thus, the human's intellect is of the same nature as divine's one. This distinction is essential since it is the one that will lead to a rational and natural solution opposed to the irrational and supernatural Newtonian solution of motion. From a logical point of view, the superiority of the Leibnizian solution is due to the distinction between two different levels allowing that a thing can be said to be "everywhere and nowhere identically" with no contradiction. This is not possible in the Newtonian framework but it will become possible in the Leibnizian one, thanks to the qualitative distinction between "essence" and "modalities of existence". To see how all this articulate with each other, let us firstly realize that if the divine character is intimately linked to "infinity", it should be remarked that the two infinities are different. The Newtonian one is associated with the continuum while the Leibnizian one is discontinuous or discrete (numbers added to each other indefinitely) as it will be seen in the forthcoming developments.

In both cases one may associate transcendence with the fact that a "transcendent entity" is one that is "everywhere and nowhere identically" which sounds as a contradiction unless one specifies what is meant by such a paradoxical expression. In Newton's case, the infinite character of velocity makes of this concept a contradictory one since the body can be everywhere simultaneously when the velocity is allowed to be infinite. Being "everywhere" also means being "nowhere" since a body is defined by its location at one specific place so that if it is here it cannot be there simultaneously. This is what Leibniz calls a miraculous event, that he does not accept in the field of physics. However, if one adopts the same words associated this time with the multiplicity of points of view, then things become clearer and no miracle is at work. The

Leibnizian multiplicity of points of view is intimately linked to a geometric progression of functions so that the ratio of the geometrical progression may be said to be “everywhere and nowhere identically” without any contradiction. It is “everywhere” in the sense that the ratio of the progression determines the place of each term of the progression but since it is a principle of action it does not belong to any one of the terms which constitute the furniture of the Leibnizian world. Here, everything is clear: the antagonistic terms “everywhere” and “nowhere” are not contradictory anymore since each term is situated at a specific level. The level of “essence” that infiltrates all the points of view (everywhere) without being identifiable to one or another point of view (nowhere). Thus, the term “everywhere” refers to “essence” while the term “nowhere” refers to “modalities of existence”. In spite of their irreproachable internal logic, these considerations developed by some of those who studied Leibniz profoundly such as C. Frémont and C. Mercer do not appear in the framework of physics until the present work. One of the main ideas of the present work is the basic distinction between two realms: “necessity” linked to “essence”, described through an “exclusive logical framework”, and “degrees of freedom” attached to “modalities of existence”, expressed by an “inclusive logical frame”. As long as one does not adopt such a distinction, there is no way to resolve some of the basic problems which are not only metaphysical but also physical.

Conclusion.

Let us conclude by recalling the distinctions between the present “multiple rationality” and the “usual” and “emergent” ones. The “usual rationality” may be characterized by its **external, simple and quantitative** characters while the “emergent one” is **internal, simple and quantitative**. Its internal character is due to its autonomy with not reference to any external kinematical framework. The present “multiple rationality” may be developed progressively starting from what is known before distinguishing between three main phases. The first phase consists in the development of an extended procedure internalizing the external “usual rationality” and unifying it to the “emergent one”. One gets then a **unified, internal, double and quantitative** formulation. The second phase consists in the discovery of a scale recurrent law that includes the two rationalities as well as other possibilities among which the “partial rationality” evoked by Taylor and Wheeler (privilege of invariant properties) without being developed in a completely autonomous manner. This second phase leads to a **unified, internal, multiple and quantitative** formulation: the different points of view on motion become ordered in a systematic manner through a recurrent sequence. The third phase consists in the development of a general qualitative procedure where the non-conserved entities do not need to be specified and determined to obtain a quantitative relation between the conserved entities. Thus, we are led to a **unified, internal, multiple semi-qualitative, semi-quantitative** formulation of the principle of relativity. It is only at this level that one really understands the intimate structure of dynamics.

APPENDICES

Summary of the Appendices: In order to distinguish the conceptual and physical information given in the main text from the mathematical and historical ones, different Appendices are proposed. **Appendix A** deals with the extended derivative and its physical justification. **Appendix B** shows the limitation obtained when symmetry requirements are imposed on the extended derivative associated with multiple scales. **Appendix C** is devoted to the role played by the extended derivative in relation to discontinuities and particularly how this extended derivative constitutes a "discontinuity absorber". **Appendix D** is concerned with some historical questions relative to Descartes dynamics as well as to the "vis viva" controversy and the role played by d'Alembert in its closure. **Appendix E** deals with the catenary's curve which played a major role in Leibniz investigations relative to the multiplicity of points of view on a given reality. **Appendix F** is devoted to the Lagrange-Hamilton formalism related to the present approach as well as to some historical confusion. **Appendix G** deals with inclusive logic and its formalisation through the inter-correlated trans-subjective and inter-subjective procedures. **Appendix H** focuses the attention on a possible relation between the problem of inelastic collisions and the deformation of a straight line interpreted as a local imprint (trunk) of a tree like structure. **Appendix I** is devoted to the different interpretations met, according to whether the attention is drawn on kinematics or dynamics. **Appendix J** recalls the usual and emergent rationalities with some comments concerning the present approach and its link to these rationalities as well as to another method proposed by Taylor and Wheeler. **Appendix K** introduces a direct link between Newtonian dynamics and its Leibnizian interpretation and extension before deriving the general differential expression of the dynamical relativity principle. **Appendix L** establishes the explicit expressions associated with the extension of Descartes dynamics and its link to "doubly special relativity". **Appendix M** is devoted to the link between the present Leibnizian framework associated with dynamical relativity and space-time physics. **Appendix N** concerns the translation and the comment of the basic points of a previous work better understood in the light of the present dynamical relativity exposition. **Appendix O** establishes a link between the usual work associated with history of science and the present formulation. **Appendix P** concerns a fruitful analogy between the oscillator problem and fundamental physics. In particular, it is shown that the passage from the non-damped oscillator to the damped one corresponds to a passage from hyperbolic to Finsler geometry. **Appendix Q** makes a net distinction between apparent and real anisotropies and links them to global and local constraints and limitations. **Appendix R** shows that in the framework that extends Newtonian and Einsteinian dynamics where parity (symmetry under reflection) is broken, the notion of velocity splits into two different concepts distinguished by this symmetry breaking. **Appendix S** provides an explicit solution associated with the trans-subjective version of the principle of dynamical relativity in its most general case. **Appendix T** classifies the different solutions that extend the Einsteinian one either through the requirement of energy finiteness or through broken parity. **Appendix U** is devoted to a comment on the process of discovery and the "principle of good things". **Appendix V** establishes a link with quantum mechanics through an extension of Klein-Gordon and Schrödinger equations where parity or symmetry reflection is broken. **Appendix W** articulates the couple "essence-existence" through the notion of a "substantial link" at the basis of Leibniz dynamics. **Appendix X** explores the possible origin of the idea of points of view in Leibniz methodology. **Appendix Y** distinguishes between three complementary ideas associated with the dynamical Leibnizian principle at the basis of this formulation: relativity, identity of indiscernibles and plenitude. Finally, **Appendix Z** is devoted to the relation between Einstein and Bergson approaches of motion.

Appendix A

Comments on the notion of a μ-derivative and justification of the form of the deviators

The "deviators" $D_\mu = D_\mu^{\mu}(v_\mu) = D^{\mu}(v_\mu , \mu)$ constitute the fundamental tool to account for the infinite multiplicity of points of view on motion. Although one may avoid the indices μ at the qualitative level, one needs them at a later step to define a principle of order between the different points of view (governed by a recurrent sequence: plenitude principle). These deviators occur through the passage from the usual derivative to the μ-derivative, when replacing the simple composition of motion v + V (initially developed by Huygens) by a multiple one:

$h_\mu(v + V) = h_\mu(v)\ T_\mu\ h_\mu(V) = v_\mu\ T_\mu\ V_\mu$, compatible with Leibniz's requirement where one performs the following change of variables: $v_\mu = h_\mu(v)$, $V_\mu = h_\mu(V)$ constructing thus a multiplicity of points of view on motion. The unique additive parameter v is replaced by the multiple one v_μ . Since these variables are not additive anymore, one is then led to the notion of a μ-derivative, which operates through $v_\mu\ T_\mu\ V_\mu$ instead of v + V. With this in mind, it becomes easy to realize that the usual derivative, a limit of the following particular linear combination:

$[f(v + V) – f(v)]/V \Leftrightarrow a\, f(v + V) + b\, f(v)$ with $a = – b = 1/V$

has to be replaced by this other linear combination:

$[f^{\mu}(v_\mu\ T_\mu\ V_\mu) – f^{\mu}(v_\mu)]/V_\mu \Leftrightarrow a_\mu\, f^{\mu}(v_\mu\ T_\mu\ V_\mu) + b_\mu\, f^{\mu}(v_\mu)$ with $a_\mu = – b_\mu = 1/ V_\mu$

This modification leads to an extended derivative, when $V_\mu \rightarrow 0$. Notice that the "+" sign occurring twice in the first linear combination (+, +), occurs only once in the second linear combination (T_μ ,+): the introduction of the new multiple operation T_μ leads to a more general class of derivatives.

In the above discussion, the idea of multiplicity of points of view was introduced starting from an additive composition of motion and extending it to an a priori infinite multiplicity of non-additive composition laws, each one associated with one point of view on motion. [In the forthcoming development we shall present another way closer to Leibniz's intuition according to which the state of rest is an accumulation point: a limit of an infinite number of curves (each constituting one point of view on motion) that converge to each other (following a unique tangent) in the vicinity of the origin. Leibniz is telling us that what is considered to be "one" by Newton may be an "infinite many" possessing a treelike structure. One does not see the whole tree simply because all the 17th century experiments are local and show only the trunk of the tree. In other words, the different composition laws of motion (tree) reduce locally to a unique additive composition law (trunk)].

In order to put this idea into a formal manner, one starts by noting that if $v_\mu' = v_\mu T_\mu V_\mu$ represents motion in a reference frame R' translated from the frame R in which v_μ is defined, then in the absence of any translation ($V_\mu \rightarrow 0_\mu = 0$), one should get $v_\mu' = v_\mu$. Thus, appears the following constraint, for any point of view:

$$V_\mu \rightarrow 0_\mu = 0 \Rightarrow v_\mu T_\mu V_\mu \rightarrow v_\mu$$

In addition to this constraint one should impose another one where the different non-additive composition laws (global form) tend to a unique additive composition law in the vicinity of the origin (local form):

$V_\mu \to 0_\mu = 0$, $v_\mu \to 0_\mu = 0 \Rightarrow v_\mu T_\mu V_\mu \to v_\mu + V_\mu \to v + V$

The above two constraints are formazlized as follows:

$v_\mu T_\mu V_\mu = v_\mu + D^\mu(v_\mu, V_\mu; \mu)V_\mu$

where the non-additive character is concentrated in the expression of $D^\mu(v_\mu, V_\mu; \mu)$. In order to satisfy the two above constraints, the expression of $D^\mu(v_\mu, V_\mu; \mu)$ cannot be arbitrary, it has to be compatible with the two following constraints:

$D^\mu(v_\mu, 0; \mu)$ should remain finite so that $D^\mu(v_\mu, V_\mu; \mu)V_\mu \to 0_\mu = 0$ when $V_\mu \to 0_\mu = 0$

and

$D^\mu(0, 0; \mu) = 1 \ \forall \mu$ so that $v_\mu T_\mu V_\mu$ reduces locally to $v_\mu + V_\mu$ or $v + V$ as explained above.

Before dealing formally with this extended derivative, let us recall that in addition to the local additive property, one may account for a global additive one considered as one point of view on motion among others. When we say that $v_\mu T_\mu V_\mu$ is non-additive, this does not necessarily exclude the existence of one additive composition of motion. Thus, the additive character appears as a necessity at the local level (the whole methodology is based on this fact), while it is facultative at the global level (a non-necessary point of view among others). This fact is of a major importance since if one assumes its necessity and uniqueness at the global level, the whole structure collapses, for in this case only one point of view survives: the additive one, which is contrary to the Leibnizian multiplicity.

Development of the extended derivative.

One starts with the following definition

$$D^\mu(v_\mu, V_\mu; \mu) = [v_\mu T_\mu V_\mu - v_\mu]/V_\mu \Leftrightarrow v_\mu T_\mu V_\mu = v_\mu + D^\mu(v_\mu, V_\mu; \mu)V_\mu \qquad \text{(A1)}$$

justified above. With such a writing, one may express the linear combination given in the beginning of this Appendix $[h^\mu(v_\mu\, T_\mu\, V_\mu) - h^\mu(v_\mu)]/V_\mu$ as follows:

$$[h^\mu(v_\mu\, T_\mu\, V\mu) - h^\mu(v_\mu)]/V_\mu = D^\mu(v_\mu, V_\mu ; T_\mu)\{[\, h^\mu(v_\mu + U_\mu) - h^\mu(v_\mu)] / U_\mu \} \qquad \text{(A2)}$$

where we have set

$$U_\mu = D^\mu(v_\mu , V_\mu ; \mu)\, V_\mu \qquad \text{(A3)}$$

On assuming that the limit condition satisfies

$$V_\mu \to 0_\mu = 0 \Rightarrow U_\mu \to 0_\mu = 0 \qquad \text{(A4)}$$

whose “raison d’être” is explained above and using compact notations

$$D_\mu = D_\mu{}^\mu (v_\mu) = D^\mu (v_\mu ; \mu) = D^\mu(v_\mu, 0; \mu) \qquad \text{(A5)}$$

one may write

$$d_\mu H/dv_\mu = [h^\mu(v_\mu\, T_\mu\, 0_\mu) - h^\mu(v_\mu)]/0_\mu = D_\mu\{ [h^\mu(v_\mu + 0_\mu) - h^\mu(v_\mu)] / 0_\mu \} = D_\mu\, dH/dv_\mu \qquad \text{(A6)}$$

The extended derivative becomes then related to the usual one through the following compact notation

$$d_\mu/dv_\mu = D_\mu\, d/dv_\mu \qquad D_\mu = D_\mu{}^\mu (v_\mu) = D^\mu (v_\mu ; \mu), \quad H = h^\mu(v_\mu) \qquad \text{(A7)}$$

Let us insist on the fact that the translation operator d_μ/dv_μ takes its source in the particular linear combination between two conserved entities: $f(v + V) - f(v)$ (finite difference) initially considered by Huygens. Leibniz’s methodology amounts to add to this procedure two ideas: (i) a passage from a finite translation to an infinitesimal one $V\to 0$ and (ii) a passage from one point of view on motion associated with the couple (v, V) to a multiplicity of points of view (v_μ , V_μ). Strictly speaking, only the first passage leads to the notion of a derivative. The second one goes beyond the derivative deforming it in an infinite number of different ways each of which constituting one different point of view on motion.

Remarkable properties.
Let us note that if one sets

$$p = d_\mu E/dv_\mu = D_\mu\, dE/dv_\mu \qquad \Leftrightarrow \qquad D_\mu = p/[dE/dv_\mu] \qquad \text{(A8)}$$

one may deduce the following first and second order differential operators

$$d_\mu/dv_\mu = D_\mu\, d/dv_\mu = p\, d/dE \qquad \text{(A9)}$$

and

$$d_\mu{}^2/dv_\mu{}^2 = \{D_\mu\, d/dv_\mu\, [D_\mu\, d/dv_\mu]\} = \{p\, d/dE\, [p\, d/dE]\} = p^2\, d^2/dE^2 + p\, (dp/dE)\, d/dE \qquad \text{(A10)}$$

These expressions are useful to account for the principle of dynamical relativity through the passage from the subjective undetermined entities v_μ [$D_\mu = D^\mu (v_\mu ; \mu)$ yet unknown] to objective ones, associated with the conserved quantities E and p. In particular, one easily shows that the application of (A10) to energy E leads to the following simplification.

$$d_\mu{}^2E/dv_\mu{}^2 = \{D_\mu\, d/dv_\mu\, [D_\mu\, d/dv_\mu]E\} = \{p\, d/dE\, [p\, dE/dE]\} = p\, (dp/dE) = p\, p' \qquad \text{(A11)}$$

so that the multiplicity of undetermined points of view disappears from the equation. One passes directly from the following complicated second order derivative

$d_\mu^2E/dv_\mu^2 = \{D_\mu\, d/dv_\mu\, [D_\mu\, d/dv_\mu]E\} = D_\mu^2\, d^2E/dv_\mu^2 + D_\mu\, (dD_\mu/dv_\mu)\, (dE/dv_\mu)$

to a first order much simpler one: $p\, dp/dE$. This will play a major role in the passage from the subjective expression of the principle of dynamical relativity to its objective or "trans-subjective" form recalling that objectivity is associated with the conserved entities while subjectivity is related to non conserved ones. The different points of view on motion are denoted by the Greek index μ and expressed through the variables v_μ.

Appendix B

Symmetry requirement and its consequence on the form of the deviators.

Appendix A performs a generalization of the usual derivative to account for a multiplicity of points of view on motion. Here, we shall make a number of physical assumptions. These apply to the deviator that we shall determine in the particular case (associated with Einstein's dynamics) where **energy is an even function while impulse is an odd one**. The account for the even and odd requirements associated with energy and impulse will lead to a restriction on the structure of the deviator. Since impulse derives from energy, on setting

$E = E_0\, f^\mu(v_\mu) = E_0\, f^\mu(-v_\mu) \quad \Leftrightarrow \quad v_\mu = f_\mu(E/E_0) \qquad f^\mu(0) = 1$ (B1)

one easily notices that only if

$D^\mu(v_\mu\, ;\, \mu) = D^\mu(-v_\mu\, ;\, \mu)$ (B2)

the impulse $p = d_\mu E/dv_\mu = D^\mu(v_\mu\, ;\, \mu)\, dE/dv_\mu$ corresponds to an odd function.

The deviators possess the same symmetry as energy. On expressing these deviators in terms of energy instead of v_μ, using (B1), and after the **separation of the continuous variable entities v_μ from the discrete ones** μ, $[D^\mu(v_\mu\, ;\, \mu) = D^\mu(v_\mu)*g(\mu)]$, one gets

$D_\mu = D^\mu(v_\mu\, ;\, \mu) = D^\mu(v_\mu)*g(\mu) = D^\mu[f_\mu(E/E_0)]*g(\mu) = D(E/E_0)*g(\mu) \qquad D = D^\mu f_\mu$ (B3)

One may consult Appendix G for more details concerning the mechanism of compensation that eliminates the multiplicity of the points of view denoted by the Greek index μ in the expression of $D = D^\mu f_\mu$.

In the forthcoming developments we shall establish the following important result

$\mathbf{D_\mu = [D(E/E_0)]^{g(\mu)} = [D(E/E_0)]^{a-\mu}}$ **(B4)** with $D(1) = 1$ so that all the curves tend locally ($E \rightarrow E_0$) to a unique tangent leading to $D_\mu = 1$ for any μ. The proof of Eq.(B4) is obtained in two steps. One recalls that impulse p should verify, for any μ and β

$$p = D_\mu\, dE/dv_\mu = D_\beta\, dE/dv_\beta \qquad \text{(B5)}$$

from which one deduces

$$D_\mu / D_\beta = dv_\mu / dv_\beta = I(E\,;\,\mu,\beta) \qquad \text{(B6)}$$

This expression possesses the following properties

$$I(E\,;\,\mu,\eta) = I(E\,;\,\mu,\beta)\; I(E\,;\,\beta,\eta) \Leftrightarrow dv_\mu / dv_\eta = (dv_\mu / dv_\beta)\,(dv_\beta / dv_\eta) \qquad \text{(B7)}$$

To verify such a requirement one sets

$$I(E\,;\,\mu,\beta) = [I(E)]^{\beta-\mu} \qquad \text{(B8)}$$

so that one is left with

$$D_\mu / D_\beta = [D(E/E_0) * g(\mu)]/[D(E/E_0) * g(\beta)] = I(E)^{\beta-\mu} \qquad \text{(B9)}$$

In order to get a solution compatible with (B6) one identifies the operation $*$ in (B3) with exponentiation and sets: $g(\mu) = a + b\mu$, leading to

$$D_\mu = D(E/E_0) * g(\mu) = D(E/E_0)]^{g(\mu)} = [D(E/E_0)]^{a+b\mu} \qquad \text{(B10)}$$

The substitution of (B10) into (B9) leads to

$$D_\mu / D_\beta = [D(E/E_0)]^{a+b\mu}/[D(E/E_0)]^{a+b\beta} = [D(E/E_0)]^{b(\mu-\beta)} = I(E)^{\beta-\mu} \qquad \text{(B11)}$$

A simple identification procedure shows that one has

$$b = -1 \Rightarrow g(\mu) = a - \mu\,, \quad \text{and} \quad I(E) = D(E/E_0) \qquad \text{(B12)}$$

The account for (B6), (B11) and (B12) yields

$$dv_\mu / dv_\beta = I(E\,;\,\mu,\beta) = [D(E/E_0)]^{(\beta-\mu)} \qquad \text{(B13)}$$

and

$$D_\mu = D^\mu(v_\mu\,;\,\mu) = D(E/E_0) * g(\mu) = [D(E/E_0)]^{g(\mu)} = [D(E/E_0)]^{a-\mu} \qquad \text{(B14)}$$

This last expression proves the validity of the above-mentioned important result given through Eq.(B4). This relation will play a major role in the determination of the infinite multiplicity of points of view on motion.

Comment on some possible links between inertia, energy and mass concepts.

Let us notice that on setting

$$D = Id \quad \Leftrightarrow \quad D(E/E_0) = E/E_0 \qquad \text{(B15)}$$

one is left with what we call the Leibnizian "microscope" function

$$D^{\mu}(v_{\mu}\,;\,\mu) = (E/E_0)^{a-\mu} \quad , \quad E/E_0 = f^{\mu}(v_{\mu}) \qquad \text{(B16)}$$

That allows to look at a same reality at different scales, depending on the values of μ with respect to a. For $\mu = a$, one gets an additive composition law associated with motion since the "microscope" function $D^{\mu}(v_{\mu}\,;\,\mu) = (E/E_0)^{a-\mu}$ reduces to unity.
The constraint given in (B15) seems natural if one adopts the Leibnizian intuition about scale laws, leading to a well-determined solution. In particular, on defining "inter-subjectivity" by $I(E;\mu,\mu+1) = dv_{\mu}/dv_{\mu+1}$, one gets the opportunity to account for inertia "I" through inter-subjectivity $I(E\,;\,\mu,\mu+1)\ \forall\mu$, up to a dimensional constant. Thus, one may define the mass concept m yet undefined through the following relation

$$I \equiv M = m\, I(E\,;\,\mu,\mu+1) = m\, dv_{\mu}/dv_{\mu+1} \qquad \text{(B17)}$$

Since there is not only one way through which the mass concept can be introduced, one may wish to perform a more intimate link with the usual way physics deals with the mass concept in its relation to impulse and motion. Thus, one may postulate that among the infinite multiplicity of points of view on motion, one of them corresponds to the proportionality relation $p = m\, v_p$, (the index p indicates proportionality). This is done in the main text where one deduces Eq.(15) instead of postulating it through the above-mentioned inter-subjectivity requirement.
Let us finally note that the substitution of (B15) into (B13) leads to $E = E_0\, dv_{\mu}/dv_{\mu+1}$ which corresponds to an **"inter-subjective" definition of energy**. Such a definition has no counterpart in "usual" and "emergent" rationalities, since each of them deals with only one point of view on motion considered to be essential. Recalling that trans-subjectivity allows one to deduce energy in terms of impulse through $pp' = E/c^2$ in Einstein's dynamics, one deduces by a simple integration $E^2 - c^2p^2 = E_0^2$, so that $E > E_0$ for all non vanishing impulses ($p \neq 0$). At this point, one easily shows that for a fixed value of E the tangent associated with the passage from a point of view to the next one is superior to unity ($dv_{\mu}/dv_{\mu+1} > 1$) so that one gets

$$dv_{\mu}/dv_{\mu+\beta} > dv_{\mu}/dv_{\mu+\beta-1} > \ldots > dv_{\mu}/dv_{\mu+2} > dv_{\mu}/dv_{\mu+1} > 1.$$

Such a progressive increase reflects faithfully the idea of looking at a thing at different scales. One may refer to the work of Parmentier [9] who shows how Leibniz establishes a link between exponentiation and the properties of differential calculus.

Final comments.

Let us emphasize the fact that the deviators would be different in a more general case, where one does not impose any symmetry requirement. Let us recall that this appendix is intimately linked to the structure of Einstein's dynamics, in view of obtaining a quantitative evaluation of the infinite multiplicity of points of view among which one will discover the velocity concept as a singular point of view among others but possessing remarkable properties as shown in Appendix F. Another singular point of view is provided by the rapidity parameter which is central because it leads to a quantitative result even if the function $D(E/E_0)$ is arbitrary. Indeed, $D_\mu = [D(E/E_0)]^{a-\mu}$ (qualitative since undetermined) reduces to unity (quantitative) for $\mu = a$ whatever the function $D(E/E_0)$. These two singular and complementary points of view on motion appear to be the most interesting ones as suggested by the "usual and emergent rationalities". The rapidity parameter constitutes the simplest and most direct point of view in dealing with the principle of relativity, since no deviation of the derivative is required (D_a= 1). As to the interest of the velocity, it lies in the methodology associated with Lagrange-Hamilton formalism, at the basis of a wide range of physical theories, and whose extension leads naturally to gauge theories for physics and symplectic geometry for mathematics. Other points of view may be underlined, such as the one mentioned above through the introduction of mass by a simple proportionality relation. This one differs from both rapidity and velocity but turns out to be natural when dealing with invariance properties as emphasized by Taylor and Wheeler as shown in Appendix M.

Appendix C

The deviator as a discontinuity absorber.

Without entering here into physical details, we shall focus the attention on the interest of the idea of undetermined multiplicity of points of view associated with a discontinuous framework.

In order to better grasp the reason for which Descartes irregular dynamical structure was not only misunderstood by 17th century physicists but also by modern ones, let us consider its internal structure in relation to dynamical relativity that leads to the following mathematics:

$$X_0 = a\,|x| \text{ and } X_n = D^nX_0/Dx^n \quad \text{such that} \quad X_1 = a\,x \quad \text{and } D/Dx = A\, d/dx \qquad x \neq 0 \qquad \text{(C1)}$$

This mathematical formal structure shows that if A = 1, then the above system becomes contradictory since one has $X_1 = a\,x$ (continuity) and for n=1, $X_1 = \pm\, a$ (discontinuity) simultaneously. However, if one lets A take any value that may depend on x [A = A(x)], the function A(x) will absorb the discontinuity associated with $X_1 = \pm\, a$ since one gets

$$X_1 = a\,x = DX_0/Dx = A(x)\, dX_0/dx = A(x)\, a\,x / |x| \;\Rightarrow\; A(x) = |x| \qquad \text{(C2)}$$

where A(x) plays the role of a discontinuity absorber.

On repeating the procedure endlessly as required by the dynamical relativity principle, one gets

$$X_{2n} = a\,|x| \qquad X_{2n+1} = a\,x \quad \forall\, n \in N \qquad \text{(C3)}$$

Here lies the main reason for which 17^{th} century scientists did not realize that Descartes dynamics was not absolutely false, although invalid at the origin and in its vicinity, as emphasized by Leibniz. Here also lies the reason for which Leibniz's methodology (associated with the multiplicity of points of view, mathematically expressible through the introduction of undetermined degrees of freedom) was misunderstood. We have here a typical example showing that Leibniz's approach seems to be contradictory only if one applies it in a too narrow framework : the one associated with $A = 1$ rejects the account for any other degree of freedom.

Let us note that in the well-known parabolic case $X_0 = a\, x^2/2$, the application of the same procedure with $A = 1$ leads to

$$X_0 = a\, x^2/2, \quad X_1 = d\, X_0/dx = a\, x, \quad X_n = d^nX_0/dx^n = a \text{ for } n = 2 \text{ and } 0 \text{ for } n > 2 \qquad \text{(C4)}$$

Thus, one is left with two equations depending on x, each of which will correspond to one conservation law. As long as one remains attached to $A = 1$ (absence of any possible multiplicity of points of view on motion) it appears that among the two examples ($X_0 = a\,|x|$ and $X_0 = a\, x^2/2$) only the parabolic one is admissible and non contradictory. In particular, the above discontinuous structure which turns out to be isomorphic to Descartes dynamics seems to be totally false, not only because of the discontinuity but also because the latter contradicts the necessity of having two conservation laws to deal with the problem of frontal elastic collision at the basis of the development of dynamics. On admitting the multiplicity of points of view on motion through the above mentioned procedure, one does not only absorb the discontinuity but also recovers two and only two conservation laws, as required by the internal logic associated with the physical problem. This fact is remarkable, since among the infinite number of possibilities very few turn out to be dynamically admissible. So, as it will be shown in the main text, in the Section dealing with prenewtonian dynamics, there is a close relationship between Einstein's dynamics and the Cartesian one. This intimate link may be explained through the following common mathematical properties

$$g(x)\, g'(x) = x \quad \text{for} \quad g(x) = |x| \qquad \text{(Descartes)} \qquad \text{(C5)}$$

$$f(x)\, f'(x) = x \quad \text{for} \quad f(x) = [1 + x^2]^{1/2} \qquad \text{(Einstein)} \qquad \text{(C6)}$$

where f and g correspond to the functions that relate energy to impulse up to multiplicative factors so that one recovers the dimensionality of each physical entity.

The main difference is that Descartes dynamics is irregular while Einstein's one is regular. This does not hold for the parabolic structure. (In Huygens-Newton structure $h(x) = 1 + x^2/2 \Rightarrow h(x)\, h'(x) \neq x$)

Relation between continuity and discontinuity in Leibniz approach of nature.

Descartes dynamics was not totally rejected by Leibniz but only partially: it may be corrected in such a way that it may finally be conciliated with the principle of continuity to which Leibniz was attached. For Leibniz, (unlike the in usual analytical models developed by physics), a discontinuity is not necessarily a monster but it may be a local singularity whose regularisation reveals a higher global continuous framework, including the discontinuity or the irregularity as a limit case. Such a limit case may be locally useful for dealing with physics at some scale. Contrary to all expectations, Descartes dynamics turns out to be much more interesting than what is usually believed. If I was sensitive to this question more than any other physicist, it is not because of a conscious philosophical position, (at least in the initial phase of my investigations), but because of the nature of my previous research intimately related to discontinuities and interfacial properties. Having worked mainly in a non-analytic framework dealing with continuous media presenting discontinuities and singular surfaces, I was particularly sensitive to Descartes dynamics, which is precisely non-analytic, so that the usual methods of conventional physics cannot be applied to such a framework. This clearly shows that the rejection of Descartes dynamics cannot be obtained by the usual methods of rational mechanics (including the one associated with the Lagrange-Hamilton formalism). A possible rational rejection of Descartes dynamics requires the development of a wider framework, capable of dealing with non-analytic functions and discontinuities, that neither d'Alembert nor any other physicist had developed since this time. Consequently, the acceptance or rejection of Descartes dynamics is an open question. Obviously, the rejection of a dynamical structure on an empirical ground is not rationally acceptable as was well noted by Leibniz. After having observed the internal structure of a drop of a biological liquid through Leeuwenhoek's microscope ("The most marvellous thing I have ever seen" wrote Leibniz), he was very sensitive to the fact that each sort of measurement is associated with a specific scale and may loose its validity at other scales. Thus, **a dynamical framework may be valid at one scale but invalid at another one. Newtonian dynamics is a famous well-known example of this fact. Descartes dynamics will turn out to be another example but to see this one should develop an appropriate methodology capable of dealing with the principle of relativity beyond the continuous and analytical framework.**

Appendix D

Various misinterpretations and misunderstandings.

a)Different appreciations of the Cartesian framework: Simple correction or Total rejection.

Physicists reject Descartes dynamics, recognizing its non analytic character and its impossibility to articulate the state of rest with that of motion as shown by Leibniz. They accept the Newtonian dynamics, while recognizing its local character since it applies only to small velocities. In order to grasp the main difference between the evaluation of this situation by physicists and by Leibniz, let us keep in mind the following argument, typically Leibnizian, and which makes a net distinction between local and global approaches. **If Descartes dynamics does not apply in the vicinity of the origin it may apply far from the origin. If this turns out to be true then, it should be put at the same level as the Newtonian dynamics, the first applying at one scale and the second at another scale.** Since no procedure is available to judge the possible local validity of Cartesian dynamics, one should not reject it, but develop an appropriate method capable of deciding whether it is "dynamically admissible" or not. What is meant here by "dynamical admissibility" is the compatibility of a dynamical framework with the principle of relativity. Thus, the main problem lies in the fact that until today, physicists deal only with "physical admissibility" in the sense that if a dynamical theory does not enter into the mould of Lagrange-Hamilton formalism it is said to be "physically inadmissible". This is the case of Descartes dynamics to which one cannot associate a Lagrangian as shown in the third part of this work. But a mechanical Lagrangian is doubly constrained, firstly by its sole application to motion in space-time through the velocity concept and secondly by its confinement to the continuum and more precisely to analytical functions. The difficulty lies in the fact that a rational evaluation of Descartes dynamics requires to go not only beyond the too narrow idea of motion through the velocity, but also beyond the continuous and analytical framework. Only then, one is able to deal rationally with the admissibility or not of Cartesian dynamics. That is why a Leibnizian framework, including an infinite multiplicity of perspectives, and apt to deal with generalized functions, constitutes a serious candidate for this task. In brief, one should emphasize that, according to Leibniz's philosophy, the Cartesian framework is not to be rejected but only corrected in such a way that it remains valid at some scale but not at the origin. (Appendix L shows the obtained solutions associated with the correction of Descartes dynamics following Leibniz methodology using integration constants to account for different scales).

b) Recall of the vis viva controversy in relation to d'Alembert's evaluation and link to the other controversy associated with the least action principle.

In closing the debate related to "vis viva" controversy, pretending that Descartes dynamics is to be rejected contrary to Huygens-Leibniz and Newton dynamics (equivalent since they propose the same solution using only different words [25]), d'Alembert fixed too tightly the aim of mechanics paving the way for what has been known since this time by space-time physics. If d'Alembert's judgement is acceptable from a **mathematical point of view**, it is not valid from a **conceptual standpoint** since the concepts behind Newton's approach and those of Leibniz are antagonistic. If these concepts seem to give the same results, it is because the parabolico-linear functions (specific of classical dynamics: $1/2mv^2$, mv) are none other than limit cases of any even and odd couples of functions, each of which being one point of view on motion. In spite of the extraordinary difference that separates Newton's world composed of one curve (parabolic) from Leibniz's one composed of a family of curves (tree like regular structures locally confused with parabolic and linear forms constituting the trunks) this difference vanishes at the limit of weak energy or motion. Leibniz's methodology was cut from its philosophical roots and adapted to presently available physics. This behaviour does not dispense justice to Leibniz's open mindedness. This is typically the case of those who try to rehabilitate the Leibnizian approach by associating it with the principle of least action (due to Maupertuis before Lagrange and Hamilton) through the controversy initiated by Koenig and followed by many scholars who did not realise the richness of Leibniz thought, confusing the "**least of all possible actions**" with the "**best of all possible worlds"**. The present analysis clearly shows that such an association is logically untenable, the first case corresponding to **simple curves** while the second dealing with **families of curves** (tree like structures).

c) Leibniz's dynamics versus Newton's one and quest for a unifying principle.

Contrary to d'Alembert's belief, the controversy between Newton disciples and Leibniz followers is not only a dispute on words, as he claimed, but on concepts. The adoption of the Leibnizian conceptual framework shows that Newton committed two deadly sins of inventing non-existent entities such as "absolute space and time" to explain concrete events, and of giving free reign to personal bias by imposing his kinematical point of view on motion through the velocity concept defined as the ratio between a length and a duration. Leibniz was right in claiming that Newton's concept, of invisible absolute space was a myth and a dangerously misleading myth. The present Leibnizian approach tends to show that the so-called "fundamental (and primary) equation of dynamics" F = m a, or its time integral p = mv is neither **fundamental and primary** nor, strictly speaking, **dynamical.** It is not a fundamental entity since it corresponds to one point of view among others. It belongs to "subjective modalities of existence" and not to "essence". It is not a primary entity but only a derived one deducible from the principle of relativity, (neglected by Newton and his followers for centuries). It is not dynamical, in a strict sense of the word, but only statico-kinematical, since it is defined by a static or invariable concept (the mass) associated with a kinematical definition of motion. For Leibniz, to whom we owe the term "dynamics", a truly dynamical approach of nature should be autonomous, and has not to rely on kinematical concepts such as the velocity. The latter is usually imposed and confirmed by Lagrange-Hamilton formalism, as being the only rational way to deal with motion, which is a false statement as emphasized by the present work. Leibniz was right to abhor the Newtonian abstractions that select only one point of view out of a multiplicity before making of it the basic substrate of the concrete world. Leibniz was among the few who were too honest and too intelligent to ignore the immediate data, and to avoid the current interpretation that does not distinguish between a thing and its shadow. This led him very far, since after having adopted the mechanistic science of his time, he had the courage to swim against the tide and get back to rediscover the relevance of Aristotle's concepts of essence and its various modalities of existence. The rediscovery of Aristotle, and its combination with the main positive contribution of modernity, actualized through the principle of relativity, led Leibniz to his monadic, architectonic universe, where the "monad" is at once an entity or perspective, living its own conscious life for itself, and at the same time the unconscious agent of some historical trend, a relatively insignificant element or perspective in the vast whole composed of a large number of perspectives. If this number is in principle infinite, it should be emphasized that only some monads have the right for a singular existence (among which the Newtonian cinematic perspective). This explains why Leibniz was at the same time in agreement with Newton, in so far as practical applications are concerned, but in a full disagreement with him as to the basic principles.

Leibniz belief in the existence of an underlying unity.

Leibniz was the least superficial of men; he could not swim with the tide without being drawn irresistibly beneath the surface to investigate the darker depths below. He could close his eyes but never forget that he was doing so. He died oppressed by the burden of the task he could not achieve, and was one of those great men who could neither reconcile nor leave unreconciled, the conflict of what there is and what ought to be. This violent contradiction between the data of experience and his deeply metaphysical belief in the **existence of a higher unity,** to which they must belong, mirrors the unresolved conflict between the reality of life and the underlying laws that governs it, although we cannot know more than a negligible portion of them. Leibniz perceived reality in its multiplicity as a collection of monads, perceptions or perspectives into which he saw with clarity and penetration scarcely ever equalled, but he believed in a vast unitary whole. No author who has ever lived has shown such power of insights into the variety of things. His genius lay in the perception of specific properties, the individual quality in virtue of which the given object is uniquely different from all others. Nevertheless he longed for a **universal**

explanatory principle or a higher unity in the apparent variety of the mutually exclusive bits and pieces which compose the furniture of the world. Like all very penetrating, very imaginative, very clear-sighted analysts who dissect or pulverise in order to reach the **indestructible core**, and justify their own annihilating activities by the belief that such a core exists, he continued to upset his rivals rickety constructions hoping that the underlying unity would presently emerge from the destruction of the different non necessary arguments, unworthy to be taken into account except at a superficial level.

Newton and Kant against Leibniz.

This program of research (existence of a higher explanatory principle) has faded away after Leibniz death, and most of his followers became more and more suspicious as to the possible existence of such a unifying principle. For them, the quest was vain, so that no core and no unifying principle would ever be discovered. The search for such an underlying substrate associated with active substance was even discarded from the realm of positive science with the distinctions brought by Kant's epistemological framework, according to which there is no reason to pursue such a quest, while the successful Newtonian empirical evidence points elsewhere. Kant's objections against the Leibnizian metaphysical quest appeared to scientist as a valuable and very reasonable solution. These objections, accompanied by the distinctions between the "transcendent" and the "transcendental", considered to be synonymous earlier, convinced most philosophers of nature and scientists. Since this time the debate, already closed a first time (scientifically) by d'Alembert, is closed a second time (philosophically) by Kant. This philosopher was much impressed by the success of Newton's approach and his followers such as Euler, (who influenced Kant in adopting the **Newtonian spatio-temporal view** rather than the substantial **Leibnizian view** with its monads or perspectives considered to be purely metaphysical and unworthy of any positive scientific treatment). Kant's reasonable objections, clearly formulated against any metaphysical or substantial underlying reality, have been well accepted and constituted the credo of scientists against any Leibnizian metaphysical tendency until today. In spite of its reasonable character and its acceptance by most scientists one should recognize that in a sense Kant missed the point, since the underlying unity of substance or dynamics looked for by Leibniz (and considered to be vain) has been discovered and placed in evidence through the present approach of the dynamical relativity principle. The latter appears in the famous mass-energy equivalence accounted for through **an infinite multiplicity of points of view** and not only through the **unique space-time point of view** (velocity) as shown historically. The adoption of such a view inherited from the Newtonian realm constituted an obstacle for centuries against the possibility of the discovery of the present underlying unity composed of an infinite number of perspectives (with four basic and singular ones, the others being more or less complicated combinations of them as shown in the main text).

On the problem of the whole and the parts.

Considered to be a ghost for centuries, the existence of the ontological order looked for by Leibniz, and lying behind a certain epistemological disorder, turns out to be a reality, possibly exhibited only if one admits that the usual analytical models are to be replaced by a more subtle language, apt to deal with an infinite number of perspectives rationally and simultaneously.

Leibniz was rejected from the realm of physics because of his belief in this flux of reality fragmented artificially by analytical science. According to Leibniz's approach of substance, the

accidental (non-necessary) concepts considered to be essential (such as the velocity or rapidity) are but names for ignorance of the underlying chains. These chains exist, whether we feel them or not, as shown by the revealed order that provides to each perspective a specific place in the whole treelike structure. The **impotence of the usual analytical method** for such a revelation is due to its selection of an **absurdly small section of a whole**, attributing everything to this arbitrarily chosen tiny section. How a global structure would be accounted for, if one starts by breaking things to arbitrary segments as done by the analytical model which operates directly into selected quantitative separated elements. Our ignorance of how things happen is not due to some inherent inaccessibility of the ultimate units, only to their multiplicity and, most of all, to our inability to see, hear, remember, record and co-ordinate enough of the available material. To render this possible one should construct what Leibniz calls "universal characteristics" a sort of artificial language capable to synthesize what the individual brain is unable to treat simultaneously. With the decomposition of things into tiny elements one takes the risk of getting too far from common sense experience. This explains why normal, simple people having tested things by long experience, uncorrupted by scientific theories, not blinded by the dust raised by the scientific authorities, can understand things better than a specialist founding his opinion on complicated calculations based on absurdly inadequate data. **Leibniz contrasts the concrete and multi-coloured reality of individual lives with the pale abstractions of a certain form of science, that takes its own empty categories for real facts**. The Newtonian and abstract material point is abhorred from Leibniz's approach, and replaced by a concrete physical one such as a grain apt to become a tree if planted in a fertile ground. Here lies the Leibnizian intuition of the state of rest associated with substance. It does not correspond to a **simple** but to an **accumulation point** that blossoms, giving birth to different stems each of them will constitute one point of view.
The reason for which Newton succeeded while Leibniz failed (in so far as dynamics is concerned) could be the fact that the first was picking flowers when the second was digging to reach the roots.

The fox and the hedgehog.
There is a line among the fragments of the Greek poet Archilochus which says: "the fox knows many things, but the hedgehog knows one thing." As pointed out by Isaiah Berlin: "scholars have differed about the correct interpretation of these dark words, which may mean no more than that the fox, for all his cunning, is defeated by the hedgehog's one defence. But taken figuratively, the words can be made to yield a sense in which they mark one of the deepest differences which divide writers and thinkers, and, may be, human beings in general. For there exists a great chasm between those, on one side, who relate everything to a central vision, one system, less or more coherent or articulate, in terms of which they understand, think and feel – a single, universal, organizing principle in terms of which alone all that they are and say has significance – and, on the other side, those who pursue many ends, often unrelated and even contradictory, connected, if at all, only in some *de facto* way, for some psychological and physiological cause, related by no moral or aesthetic principle…. The first kind of intellectual and artistic personality belongs to the hedgehogs, the second to the foxes". If some great men can be attached to the first category or the second, when we come to Leibniz and ask to which category he belongs, whether he is a **monist or a pluralist**, whether his vision is "**one" or "many"**, whether he is of a **single substance** or compounded of **heterogeneous elements**, there is no clear or immediate answer. The question does not seem, somehow, wholly appropriate; it seems to breed more darkness than it dispels.

Here lies the main reason for which scholars differ radically as to the interpretation of Leibniz philosophy. In some respects Leibniz is known to be a pluralist with his infinity of possible worlds and the unlimited number of points of view on each of them. In other respects, he is considered to be very close to Spinoza's monism, especially because of the critics addressed to Descartes arbitrary degrees of freedom associated with God's will, that Leibniz refuses in favour of the best possible world. A number of controversies concerning Leibniz approach of nature are a direct consequence of the fact that his architectonical and monadic framework lies beyond these two categories. Those who deal with the narrow framework of 17^{th} century dynamics do not encounter the idea of infinite multiplicity of possible worlds, and of degrees of freedom associated with each of them. One is tempted then to make a net distinction between his philosophy and dynamics, to which Leibniz actively contributed, leading thus to some contradiction since Leibniz spent his life emphasizing that his philosophy and his dynamics are intimately related and should not be studied separately. The problem lies in the fact that **Leibniz's effective work on dynamics is only a germ that needs to be developed in order to reveal its potentialities**. As long as one remains confined to 17^{th} century dynamics there is no way to understand Leibniz multiplicity of points of view since its local character does not allow to reveal any multiplicity of perspectives, locally, fused together leading to the belief of the existence of only one entity associated with motion: the velocity. The present work avoids the above mentioned "contradiction" between the pluralistic Leibnizian philosophy and his apparent monistic dynamics, a germ that requires to be developed leading thus to a complete agreement between the general philosophical Leibnizian ideas and their manifestations in science through the question of motion at the basis of dynamics, (skeleton of physical reality).

APPENDIX E

The Leibnizian approach of the catenary's curve.

Let us note that the passage from the Newtonian parabolic world to the Einsteinian hyperbolic one is structurally similar to the passage from the Galilean parabolic solution (first quantitative attempt) associated with the catenary's curve to the hyperbolic one (correct solution). The final solution was discovered simultaneously by Huygens, Bernouilli and Leibniz each one using a different way or parameterization. This explains the possible transfer of some intuitive ideas expressed in the 17^{th} century, partly formalized, but never developed on a rational ground as suggested by Leibniz. Indeed, on looking for a possible hidden order lying behind the three ways of looking at a hyperbola $Z^2 - X^2 = a^2$, through different parameters x, y and z as follows

$$X = a\,x = a \sinh y = a \tan z \qquad Z = a\,[1 + x^2]^{1/2} = a \cosh y = a \sec z \tag{E1}$$

where y and z are the parameters favoured by Leibniz and Huygens respectively. A simple calculation shows that one has

$$dX/dx = a = a\,Y^0 \qquad dX/dy = aY = a\,Y^1 \qquad dX/dz = a\,Y^2 \tag{E2}$$

where we have set

$$Y = Z/a \tag{E3}$$

This highly suggests looking for other possible parameters by setting

$$dX/dx_\mu = a\, Y^{\mu - 1} \tag{E4}$$

On applying (E4) to k = n and k = m one deduces

$$dx_\eta / dx_\mu = Y^{\mu-\eta} = (Z/a)^{\mu-\eta} \tag{E5}$$

so that one may include x, y and z in such a way that they correspond to x_1, x_2 and x_3 respectively. Here lies the order that Leibniz have possibly discovered through his various investigations in differential calculus associated with the catenary's curve and more generally with the study of conics following Pascal's work. ***Pascal introduced the distinction between the geometrical mind and the subtle one. The subtlety lies here in the fact that one may look at one object (here the hyperbola) from different points of view each of them possessing its specific properties***. In addition, let us focus the attention on the fact that functions such as ArgtanhX and ArctanX were not classified as today at Leibniz epoch but they appeared only through their integral forms $\int dX/(1+X^2)^{1/2}$ and $\int dX/(1+X^2)$. These forms suggest the following extension $\int dX/(1+X^2)^{\mu/2}$ where μ may take an infinite number of values. As shown by Parmentier in Ref [9], Leibniz was much interested in such integral forms, as well as in their development in series. This led him not only to the idea of infinite multiplicity of points of view, but also to realize that both Huygens an Descartes dynamics might be an imprint of a higher dynamics capable of unifying the two different and local frameworks into a unique global one. These mathematical discoveries are, in our opinion, at the basis of **Leibniz's optimism** as to the possibility of constructing **a general framework conciliating the old problem of the "One" with the "many"** inherited from Plato and Aristotle philosophies. According to Leibniz, the "One" or the basic substance is not to be regarded as a thing but as a principle apt to generate an infinite number of things. At different occasions, Leibniz gave the example of a geometric progression as an example where the "One" is exemplified by the geometrical ratio of the progression while the different "living atoms", "monads", "perceptions" or "perspectives" are constituted by the infinite number of the elements generated through an endless repetitive procedure. This is precisely the case in (E4) as well as in (B13) subject to (B15) except that one is dealing with infinitesimal variables instead of pure numbers which is usually the case in elementary geometrical progressions.

Appendix F

Link to Lagrange-Hamilton formalism and necessity of other points of view on motion.

a) Lagrange-Hamilton formalism related to Leibniz's approach.

If one adopts non dimensional notations, the second order differential equation associated with the subjective dynamical relativity principle (5) takes on the following form

$$I^* = d_\mu p^*/dv^* = d_\mu{}^2E^*/dv^{*2} = \{(E^*)^{-2} d/dv^*[(E^*)^{-2}dE^*/dv_\mu{}^*]\} = E^* \qquad \text{(F1)}$$

where we have set

$$\mu = a+2, \quad v^* = v_{a+2}{}^* \quad v^* = v/c \quad E^* = E/E_0 \quad p^* = cp/E_0 \qquad \text{(F2)}$$

On performing the following change of variable

$$L^* = -1/E^* \qquad \text{(F3)}$$

the above second order differential equation given in Eq.(F1) and the expression of impulse $p^* = d_\mu E^*/dv^* = (E^*)^{-2} dE^*/dv^*$ simplify leading to

$$d^2L^*/dv^{*2} + 1/L^{*3} = 0 \qquad \text{(F4)}$$

and

$$p^* = dL^*/dv^* \qquad \text{(F5)}$$

where one recognizes the definition of impulse as the derivative of the Lagrangian with respect to the velocity.

The integration of (F4) leads to the following solution

$$L^* = -[1 - v^{*2}]^{1/2} \qquad \text{(F6)}$$

which corresponds to the Lagrangian of Einstein's dynamics where the constants of integration have been chosen in such a way that for weak velocities one gets the Newtonian parabolic form $L^* = ½ v^{*2} - 1 \Leftrightarrow L = T - V = ½ mv^2 - E_0$ ($mc^2 = E_0$). In order to distinguish the present approach from the usual one, let us recall that historically Eq.(F6) is rooted in Lorentz kinematics and given by Poincaré in Ref.[27].

The substitution of (F6) into (F5) yields

$$p^* = dL^*/dv^* = v^*/[1 - v^{*2}]^{1/2} \qquad \text{(F7)}$$

so that its combination with (F3) leads to the two following remarkable properties

$$E^{*2} - L^{*2} = p^{*2} + v^{*2} \qquad E^* - L^* = p^*/v^* + v^*/p^* \qquad \text{(F8)}$$

discussed in the second part of this Appendix and whose ratio gives the following fundamental relation associated with Lagrange-Hamilton formalism

$$E^* + L^* = p^*v^* \qquad \text{(F9)}$$

usually expressed as follows:

$$E^* = v^*\, dL^*/dv^* - L^* \qquad \text{(F10)}$$

where (F5) has been accounted for.
One recognizes here through (F5) and (F10), the general structure of the Lagrange-Hamilton formalism: the knowledge of the Lagrangian is sufficient to deduce the two conservation equations known as impulse and energy. Before dealing with the remarkable properties associated with Lagrange-Hamilton formalism and their link to prenewtonian dynamics, it is worth recalling that the Lagrange-Hamilton formalism appears to be one point of view among others and that the Lagrangian which usually determines the dynamical framework is here intimately linked to the "dynamical relativity principle" through a change of variable which simplifies the expression of this principle as shown in the passage from (F1) to (F4).

b) Historical confusions and link to conics.
In this second part, we shall focus the attention on some historical confusions and remarkable properties associated with the Lagrange-Hamilton formalism, before establishing a link with the "trans-subjective" version of the principle of dynamical relativity.
Eq.(F9) shows that the Lagrangian is to energy what the velocity is to impulse in so far as symmetries are concerned. Indeed, symmetry reflection $v^* \rightarrow -v^*$ implies $p^* \rightarrow -p^*$ while E^* and L^* remain invariant. This invariance is here ensured by the multiplication operation through p^*v^*. In 17th century physics and in the absence of any potential consideration which is the case when dealing with the problem of elastic collisions, (F9) clearly shows that when dealing with the Newtonian framework: $p^* = v^*$ and $E^* = L^* = T^* \Leftrightarrow p = mv$ and $E = L = T$, one gets the well-known parabolic relation: $T^* = ½\, v^{*2} = 1/2\, p^{*2}$. Such relations were misunderstood in 17th century physics and led to confusion. Notice also that when potential energy is not neglected ($E = T + V$, $L = T - V$ or equivalently $E^* = T^* + 1$, $L^* = T^* - 1$) then, Eq.(F9) reveals a certain mechanism of compensation since it remains invariant with or without potential energy. This is obviously not the case of Eqs.(F8) that reduce to:
$E^{*2} - L^{*2} = 2\, p^{*2} = 2\, v^{*2}$ and $E^* - L^* = 2$ when $p^* = v^*$. Here, $E^* \neq L^*$ and is compatible with $E^* = T^* + 1$, $L^* = T^* - 1$ which is the case in the Newtonian framework when potential energy is taken into account.

Having recalled the confusions met by 17th century physics before the advent of the works of Lagrange and Hamilton, let us now focus the attention on some general properties which turn out to be intimately linked to conics through hyperbolic and elliptic functions. Even if these functions appeared in dynamics only in 20th century dynamics with the advent of Einstein's approach, Leibniz made several significant remarks on such functions, in relation with mathematics and mechanical curves such as the one associated with the catenary (to which Appendix E is devoted). These remarks may be useful for a better understanding of the structure of dynamics. To this end, let us firstly note that the combination of Eq.(F8) with Eq.(F6) leads to the following remarkable hyperbolic and elliptic properties

$$E^{*2} - p^{*2} = 1 , \qquad L^{*2} + v^{*2} = 1 \qquad \text{(F11)}$$

where E* is to p* what L* is to v* except for the sign. In the 17th century, Leibniz was much interested in such forms intimately related to conics. In particular, at different occasions he invited us not to forget that a parabola may be regarded as an ellipse, one of its foci being cast to infinity. He also drew the attention on the existence of a unique differential equation whose solutions may be associated with a parabola, a hyperbola or an ellipse. The difference lies in the way one deals with the limit conditions as shown in the next paragraph.

d) Symmetry properties and link to the dynamical relativity principle.

Noting the perfect symmetry between L* and v* in (F11) then inverting L* with v* keeps the solution invariant. Thus, the following differential equation

$$d^2v^*/dL^{*2} + 1/v^{*\,3} = 0 \qquad \text{(F12)}$$

deduced from (F4) is also compatible with (F11). Note also that on replacing v* by p* on the one hand and L* by E* on the other hand, then the above second order differential equation leads to

$$d^2p^*/dE^{*2} + 1/p^{*\,3} = 0 \qquad \text{(F13)}$$

The integration of such an equation is compatible with the following hyperbolic structure

$$E^{*2} - p^{*2} = 1 \;\; \Rightarrow E^* = [1 + p^{*2}]^{1/2} \qquad \text{(F14)}$$

As well as with the parabolic one

$$E^* = 1 + p^{*2}/2 \qquad \text{(F15)}$$

The difference lies again in the choice of the limit conditions. In order to make a link with the dynamical relativity principle in its “trans-subjective” version, let us note that the second order differential Eq.(F13) may be written as follows

$$p^{*3}\, p^{*\prime\prime} = -1 \qquad\qquad p^{*\prime} = dp^*/dE^* \qquad \text{(F16)}$$

Its derivative leads to

$$[p^*\, p^{*\prime\prime\prime} + 3\, p^{*\prime}\, p^{*\prime\prime}]\, p^{*2} = 0 \qquad \text{(F17)}$$

so that one may perform a direct link with the expression given in (4, c).

Let us note that the natural parameters associated with p* and v* are

$$p^* = \sinh\varphi \qquad v^* = \sin\theta \tag{F18}$$

because of the forms given in (F14) and (F6) that lead to

$$E^* = \cosh\varphi \qquad L^* = -\cos\theta \tag{F19}$$

Noting that the combination of (F3) with (F9) leads to

$$L^*E^* + 1 = 0, \qquad p^*/E^* - v^* = 0, \qquad p^* + v^*/L^* = 0 \tag{F20}$$

One deduces the following links between the two parameters as follows

$$\cosh\varphi \cos\theta - 1 = 0, \qquad \tanh\varphi - \sin\theta = 0, \qquad \sinh\varphi - \tan\theta = 0 \tag{F21}$$

These hyperbolic and trigonometric angles associated with the Hamiltonian E* (or energy) and the Lagrangian L* are none other than the two points of view on motion relative to the orders 2 and 3 respectively since one may show that these correspond to the following relations:

$$\varphi = x_2 = v_2/c \qquad \theta = x_3 = v_3/c \tag{F22}$$

when they are compared to the full structure associated with Einsteinian dynamics developed in the first part of this work.

e) Limit of the velocity concept and necessity of other points of view on motion.

The Lagrange-Hamilton formalism is based on the velocity concept which corresponds, as well-known from Einstein's dynamics, to a bounded or finite entity. The aim here is to show that the **multiplicity of points of view is not a question of choice but of necessity**. Contrary to what one may think at first sight, even if only one point of view is sufficient to deal with a practical problem, the question of multiplicity of points of view remains an essential fact. If one deals only with the velocity concept (as usually done in conventional physics), the velocity remaining finite and possessing an asymptotical behaviour, for very high energies this parameter becomes extremely close to the asymptotical line, and therefore practically useless since no difference is measurable any more. One is then obliged to adopt another point of view on motion. This argument is purely logical and does not need any recourse to experiment in order to show it. This is the kind of logic that led Leibniz to the construction of a general inclusive framework, capable to deal with different points of view simultaneously. According to Jammer [23], Huygens wrote a letter to Leibniz in which he asserts that in a second edition of the "principia" Newton would certainly correct his error as to the infinite character of the velocity. After having been convinced of the logical difficulty associated with the possibility of an infinite velocity in principle, (an object may be encountered at different locations at the same time), most of the Newtonian scientists forgot this problem when they realised the efficiency of the Newtonian system in predicting correctly a non negligible number of events. Obviously, the necessity of adopting different points of view on motion does not lead automatically to the necessity of an infinite number of such points of view. On the contrary, the very idea of an infinite multiplicity is physically inadmissible. But Leibniz discovered, when dealing with geometrical series of functions, that even if in principle one assumes an, a priori infinite number of ordered points of view, it turns out that the calculations lead to a finite well-determined number. Only a few numbers of points of view are singular and basic, the others being too complicated to have any practical relevance: they are formed by different iterative combinations of the few basic and elementary points of view. A somehow suggestive, (although partial) analogy may be borrowed from everyday experience. One may think of a simple observation of a statue from different sides. Even if one may choose an infinite number of locations in principle, in practice four locations: east, west, north and south are sufficient to have a somehow precise idea of its different facets. This intuition is found in Leibniz writings at many different occasions and may have played a major role in his philosophy as well as in some of his mathematical discoveries.

APPENDIX G

Inclusive logic and formal implications: trans-subjectivity and inter-subjectivity.

The structures of dynamical models in their different analytical versions operate on three basic entities, two of which being conserved (energy and impulse) while the third, associated with motion (velocity or rapidity) is not. The situation is different in Leibniz's approach: he aims at the construction of a wider theoretical framework including different possible models, each of them corresponding to one specific point of view on motion. In this generalized framework, the third non conserved entity (say x), is to be replaced by x_μ where the Greek index μ accounts for the multiplicity of points of view on motion. This multiplicity leads to the introduction of an inclusive logical framework, absent from conventional dynamical models. In particular, in order to perform some necessary distinctions not needed in classical models, we associate the idea of objectivity to conserved entities Y and X and that of subjectivity to the different points of view on motion which are not conserved. With these distinctions, one is led naturally to inter-subjectivity when the subjective points of view are correlated to each other, and to trans-subjectivity when the different subjective points of view are eliminated in favour of the objective entities. In order to simplify the presentation, we shall not mix the physical and the structural elements. This appendix is only devoted to the way **trans-subjectivity and inter-subjectivity** occurs on a **purely formal level**. The physical principles behind this structure are given in the main text. The two important points that we wish to underline here (absent from usual models), concern the formal mechanism that leads to trans-subjectivity and inter-subjectivity. The basic element in the Leibnizian approach is the existence of a specific operator that generates automatically conservation law. This operator is a sort of extended derivative that may be expressed as follows:

$$O(Z) = O_{\mu\mu}(Z) = d_\mu Z/dx_\mu = D^\mu(x_\mu;\ \mu)\, dZ/dx_\mu \quad \forall\mu \qquad \text{(G1)}$$

where Z is a function of the subjective variables x_μ corresponding to Y and/or X. For reasons we do not evoke here, one is led to the following relation where the objective elements X and Y are mixed with the subjective ones x_μ and x_β through the following ratio

$$O(Y)/O(X) = O_{\mu\mu}(Y)/O_{\beta\beta}(X) = \{D^{\mu}(x_{\mu};\mu)dY/dx_{\mu}\}/\{D^{\beta}(x_{\beta};\beta)\, dX/dx_{\beta}\} = F[Y, X; \mu,\beta] \qquad (G2)$$

where Y and X depend on x_{μ} and x_{β} as follows

$$Y = f^{\mu}(x_{\mu}) \, , \; X = g^{\beta}(x_{\beta}) \qquad (G3)$$

Trans-subjective procedure.
This qualitative and underdetermined structure given in (G2), shows that when $\mu = \beta$, the different points of view expressed through the operator $O_{\mu\mu}$ are cancelled, keeping only the objective structure of dynamics associated with X and Y . One is then left with the following differential form

$$dY/dX = F[Y,X] \qquad \text{with} \qquad F[Y,X\,;\mu,\mu] = F[Y,X\,;\beta,\beta] = F[Y,X] \qquad (G4)$$

Notice that if F is well-determined by some specific requirement, then without any knowledge of the properties associated with the points of view, one obtains nevertheless a quantitative expression of (G4), which turns out to account for the **trans-subjective version** of the principle of **dynamical relativity**. Here lies the essence of trans-subjectivity where one clearly shows that the specification of the particular properties of the points of view is not of any relevance here. **The multiplicity of the subjective points of view x_{μ} disappears in favour of the objective quantities Y and X.**

The resolution of (G4) leads to a well determined relation between Y and X. This procedure consists in obtaining the objective structure of dynamics by means of the trans-subjective procedure.

Inter-subjective procedure.
In addition to the fact that the two conserved entities X and Y may be linked to each other quantitatively without specification of any point of view, in the forthcoming developments we shall place in evidence the possibility of defining the conserved entities not only as usual through a dependence with respect to one point of view, but through a combination of different points of view. To this end, we keep $\mu\neq\beta$ and we assume the existence of a relation between X and Y as follows:

$$X = R(Y) \qquad (G5)$$

Thus, one deduces from (G3)

$$X = R[f^{\mu}(x_{\mu})] = h^{\mu}(x_{\mu}) = g^{\beta}(x_{\beta}) \Rightarrow x_{\mu} = h_{\mu}[g^{\beta}(x_{\beta})] = G_{\mu}{}^{\beta}(x_{\beta}) \, , \qquad h^{\mu}\, h_{\mu} = Id \qquad (G6)$$

The inversion of the functions given in (G3) allows one to write

$$x_{\mu} = f_{\mu}(Y) \qquad x_{\beta} = g_{\beta}[R(Y)] = k_{\beta}(Y) \qquad (G7)$$

so that $D^{\mu}(x_{\mu};\mu)$ may be expressed in terms of Y and μ as follows

$D^\mu(x_\mu; \mu) = D^\mu[f_\mu(Y); \mu] = K(Y ; \mu)$ $\quad\quad$ $K= D^\mu f_\mu$ $\quad\quad$ (G8,a)

This may also be written as

$D^\mu(x_\mu) * h(\mu) = D^\mu[f_\mu(Y)] * h(\mu) = K(Y) * h(\mu),$ $\quad\quad$ (G8,b)

if one separates the continuous variables x_μ from the discrete ones associated with μ. It should also be noted that there is an implicit assumption hidden through $D^\mu f_\mu = K$, where a mechanism of compensation is at work so that the resulting function K does not depend on the discrete elements μ.To fix ideas, one may give the following example

$K(x) = D^1[f_1(x)] = D^2[f_2(x)] = D^3[f_3(x)] = \ldots\ldots$

$[1 + x^2]^{1/2} = \cosh[\operatorname{arcsinh}(x)] = \sec[\arctan(x)] = [1 - x^2/(1 + x^2)]^{-1/2} = \ldots\ldots.$

where a function may be decomposed in different manners each of which revealing a number of properties and hiding others.

Obviously, all what has been said concerning $D^\mu(x_\mu; \mu)$ holds for $D^\beta(x_\beta; \beta)$ so that one gets

$D^\beta(x_\beta; \beta) = L(Y; \beta)$ $\quad\quad$ (G8, c)

Thus, (G2) subject to (G5) and (G8) leads to the following form

$F[Y, R(Y) ; \mu,\beta] /R'(Y) = \{ K(Y ; \mu)dx_\beta\}/ \{ L(Y; \beta)dx_\mu\}$, $R'(Y) = dX/dY$ $\quad\quad$ (G9, a)

from which one deduces

$dx_\beta /dx_\mu = H(Y; \mu, \beta) = H_\beta{}^\mu(Y)$ $\quad\quad$ (G9, b)

so that one is left with the following relations:

$Y = H_\mu{}^\beta(dx_\beta /dx_\mu)$ $\quad\quad$ $H_\mu{}^\beta H_\beta{}^\mu = Id$ $\quad\quad$ (G10)

and

$X = R(Y) = R[H_\mu{}^\beta(dx_\beta /dx_\mu)] = K_\mu{}^\beta(dx_\beta /dx_\mu)$ $\quad\quad$ (G11)

The expressions of Y and X correspond to what we call inter-subjective definitions of the objective conservation laws. Let us emphasize the fact that such inter-subjective measures have no existence in conventional analytical models since these do not take into account more than one point of view on which all the rest depends.

Appendix H

From the linear relation to a tree like structure: Link to dynamics (inelastic collisions).
We propose the development of a procedure showing that the hyperbolic structure may be interpreted as the best of all possible worlds and may be regarded under an infinite number of points of view. In addition, we place in evidence the efficiency of mathematics in dealing with an old philosophical problem: association of the "one" with the "many" in a rational manner, in relation with the Aristotelico-Leibnizian paradigm.

Greek sailors realized that the form and the limit of the earth did not coincide with the horizon of observations. Later on, the scientific enterprise could be erected on a rational ground when the majority of scientists accepted to go beyond the limitations inherent to observation and experimentation. However, the limit of the world does not coincide with the limit of the language used to express it. Here lies the main reason for which it is difficult to deal with dynamics in a sufficiently precise manner. According to Leibniz's approach, the analytical language used in dynamics is inappropriate, since it does not fit with the basic requirements of this science. It is possible to deal with the principle of relativity, (the basis of dynamics), in different manners and under various points of view, but the language used in this framework is not adapted to such a multiplicity. In the 17th century, Leibniz proposed to construct an appropriate language, apt to include the richness of such a framework, a "universal language" capable of a decisional power concerning the validity of a given proposition. As long as such a proper formalisation is not specified one should do without it, taking the risk of ambiguities and contradictions.

Let us get closer to the logical and formal structure that has to be constructed. The basic starting point runs as follows: if what is assumed to be "one" is "many", then any analysis based on such a "oneness" can only be partial, grasping only some aspects of things. More precisely, the essence of motion and its vanishing limit (the state of rest) in a fixed reference frame is not "one" but "many". Thus, **the state of rest does not correspond to one point whose variation or change leads to a line but it corresponds to an accumulation point associated with a multiplicity of lines**. This "many" or this multiplicity of lines that converge to a unique point (state of rest) is not arbitrary since the convergence is obtained following a unique direction, so that locally an infinite multiplicity of lines coincide getting the impression that there is only one line. This explains why one may obtain local valid physical results, in spite of the confusion between the "one" and the "many", locally indistinguishable. Having paved the way for the construction of such a formal language, we shall start by constructing a system of differential equations including an infinite number of indistinguishable lines before deforming it in a "natural" way changing only one element and keeping all the other elements invariable. As pointed out in the main text, such a replacement of the "one" by the "many" corresponds to a return to a paradigm of the Aristotelian type, according to which substance can be accounted for through different modalities of existence (as emphasized by Leibniz who followed the mechanistic philosophy for a while before realizing its weakness and strong limitations as compared to the richer Aristotelian paradigm that needs to be adapted to the principle of relativity unknown in antiquity).

It is worth noting that Leibniz followed Aristotle in another direction, the one associated with the principle of analogy (a/b = c/d) adapted to differential calculus. The present Leibnizian construction consists in proposing that the ratio between infinitesimal entities and finite ones (or equivalently the ratio of local and global extensions) are equal. This is one of the different ways Leibniz conceives the construction of a straight line. Its formal structure corresponds to $dy/dx = y/x$ where one recognizes the Aristotelian analogy principle to which one adds infinitesimal considerations (unknown at the epoch of Aristotle) to which Leibniz brought a substantial development. Starting from this idea we shall proceed in two steps. We first develop an iterative differential procedure that leads to an infinite number of straight lines fused together then, we perform a "natural" deformation of one element transforming thus the indiscernible straight lines into discernible curves which coincide locally in the vicinity of the origin. To this end, we start by constructing an iterative procedure associated with an invariant measure as follows:

$$I = I(X) = dx_\mu / dx_{\mu+1} \qquad x_\mu = f_\mu (X) \quad \forall\, \mu \qquad \text{(H1)}$$

with the particular condition associated with $\mu = p$ (proportionality) where one imposes

$$x_p = f_p (X) = A X \qquad \text{(H2)}$$

as well as the following limit conditions

$$X \to 0 \quad f_\mu (0) \to 0 \qquad \forall\, \mu \quad I(0) \to 1 \qquad \text{(H3)}$$

The Greek index μ corresponds to an element that may take any value while the Latin index p corresponds to a particular (fixed) one. The substitution of (H2) into (H1) leads to

$$I = I(X) = AdX / dx_{p+1} \quad \Leftrightarrow \quad x_{p+1} = A\int dX / I(X) \qquad \text{(H4)}$$

so that the knowledge of I(X) allows to derive the infinite number of elements x_μ by successive iterations obtained through (H1). However, as mentioned earlier we have to deal only with ratios and derive I(X) imposing on it a **constraint that leads to an infinite number of straight lines fused together before deforming these lines leading to an infinite number of distinguished curves**. The first step consists in imposing the following constraint

$$I = x_p / x_{p+1} \qquad \text{(H5)}$$

whose combination with (H1), H(2) and (H3) leads to an infinite number of identical straight lines

$$AX = x_p = x_{p+1} = \ldots = x_\mu \quad \forall\, \mu \qquad \text{(H6)}$$

with

$$I = I(X) = 1 \qquad \text{(H7)}$$

Setting

$I = M/m \qquad x_\mu = v_\mu/c \qquad X = p/mc$

One gets

$P = m\, v_p = mv_{p+1} = mv_{p+2} \ldots\ldots \qquad M = m$

This Leibnizian interpretation possesses an infinite number of potentialities absent from the usual Newtonian scheme associated with the two conservation laws corresponding to inelastic collisions, where the same structure holds except for the multiplicity which is here degenerate since all the points of view are identical. Notice that the constant A could be identified with unity without lack of generality (A = 1). It does not play any relevant role here, since the identity between two non dimensional entities turns out to be equivalent to a proportionality relation between dimensional entities as shown in the above equations. Let us recall that, here, the main point consists in showing the consequence of a simple deformation of (H5) relative to linearity. Having developed things from both a structural and physical standpoints, let us now deform (H5) as follows:

$$I = I(x; k) = x_p / x_{p+k} \qquad k>1 \qquad \text{(H8)}$$

keeping everything else invariable. On replacing p + 1 by p + k one introduces an infinite number of other possibilities through the index k so that I(X) becomes a particular case of the new definition I(X ; k) corresponding to I(X) = I(X ; 1).

In order to get a differential equation between x_p and x_{p+k} one uses (H1) from which one deduces

$$I^k = dx_\mu / dx_{\mu+k} \quad \forall\mu \qquad \text{(H9)}$$

The combination of (H9) subject to $\mu = p$ with (H8) leads to

$$[x_p / x_{p+k}]^k = dx_p / dx_{p+k} \qquad \text{(H10)}$$

from which one gets

$$I = I(X ; k) = x_p / x_{p+k} = X / x_{p+k} = [\, 1 + C_k\, X^{k-1}]^{\,1/(k-1)} \qquad \text{(H11)}$$

We have set A = 1 without lack of generality as shown above. Having deduced I(X ; k) one may obtain the infinite number of $x_{p+\mu}$ by use of the following property

$$I^\mu = dx_p / dx_{p+\mu} = dX / dx_{p+\mu} \qquad \text{(H12)}$$

derived in (H9) subject to (H2) and (H11) from which one gets

$$x_{p+\mu} = \int dX / I^\mu = \int dX / [\, 1 + C_k\, X^{k-1}]^{\,\mu/(k-1)} \qquad \text{(H13)}$$

Notice that for small X H(13) reduces to (H6), so that the infinite number of curves coincide locally with a straight line : **none other than the tangent, common to all the curves at the origin**. In order to make a link with the structure of dynamics, we choose p = 1 (no lack of generality) and we consider the particular cases k = 3 and $C_3 = 1$. One is then left with

$$x_\eta = \int dX / I^{\eta-1} = \int dX / [1 + X^2]^{(\eta-1)/2} = \int [1 + X^2]^{(1-\eta)/2} dX \qquad \text{(H14)}$$

corresponding to Eq.(14) in the main text. Developing Eq.(H14) for η = 1, 2, 3, 4, and inverting the four basic results: $x_1 = X$, $x_2 = \text{Argsinh}X$, $x_3 = \text{Arctan}X$ and $x_4 = X/[1+X^2]^{1/2}$ (the others being obtained through a recurrent series not developed here) we find

$$X = x_1 = \sinh x_2 = \tan x_3 = x_4 / [1-x_4^2]^{1/2} \qquad \text{(H15)}$$

$$I = [1+x_1^2]^{1/2} = \cosh x_2 = \sec x_3 = 1 / [1-x_4^2]^{1/2} \qquad \text{(H16)}$$

The elimination of any x_η leads to the same hyperbolic structure

$$I^2 - X^2 = 1 \qquad \text{(H17)}$$

(The particular case k = 3 is analysed in the second part of this Appendix. It is shown that it corresponds to what Leibniz calls the "best" of all possible worlds).

The hyperbolic structure as the best of an infinite number of other possible structures.
Dealing with the catenary's curve, Leibniz focussed the attention on its relation to the hyperbolic curve and to the multiplicity of manners in dealing with it (as shown in Appendix E). It should be emphasized that Leibniz also embedded the hyperbolic relation into a more general structure as follows: (see Ref.[9]).

$$Y = [1 + X^q]^{1/q} \quad \text{or} \quad Y^q - X^q = 1 \qquad Y > 0 \qquad \text{(H18)}$$

where q is an even number such that ($q \geq 2$). Thus, q = 2 corresponds to the hyperbolic curve. Notice the analogy with Eq.(H11) associated with the deformation of linearity. The interest of this form lies in the fact that, among the infinite multiplicity of possible curves the one associated with the minimal even number is the richest in structures. To see this, let us show that the first selection consists in noting that the passage from q = n to $q = n - m \geq 2$ is richer than the passage to q = n + m. In particular, for n and m sufficiently great such that n – m = q…6, 4, 2 then one gets:

$Y \to 1$ for $X^2 < 1$ and $Y \to |X|$ for $X^2 > 1$ for p = n + m

and

$Y = [1 + X^q]^{1/q}$ … $Y = [1 + X^6]^{1/6}$, $Y = [1 + X^4]^{1/4}$, $Y = [1 + X^2]^{1/2}$ for p = n – m.

One easily notices that in the first cases the curves correspond to straight lines (quasi-broken at the following limit: $X^2 \rightarrow 1$) with no possible measurable variations. Thus, if one has to choose between the two extremes, then the lowest one is the richest since it allows getting the biggest number of measurable quantities. (One observes that the "least" number is the "best". If the "least" and the "best" may be associated with one another one should notice that, here, the "least" has nothing to do with the "least action principle"). Notice that the least number operates here on possible worlds from which one selects the best. It corresponds to the hyperbolic world that can be parameterized in different manners each of them constituting one point of view as shown in (H15)-(H16). This is in total agreement with the Leibnizian assertion as to the existence of an infinite multiplicity of possible worlds, from which one may select the best. In addition to this, one should keep in mind this second complementary assertion associated with the existence of an infinite multiplicity of points of view. This multiplicity has been studied in the case of the catenary's curve which is intimately related to the hyperbolic structure as shown (in Appendix E). These are the mathematical considerations corresponding to Leibniz metaphysical assumptions. At different occasions, Leibniz emphasized the direct link between his mathematical discoveries and metaphysical assertions.

Those who associate the "best of all possible worlds" with the "least of possible actions" at the basis of the Lagrange-Hamilton formalism make two kinds of error. Firstly, the principle of least action is not associated with structural richness, since Newtonian dynamics (being the poorest of all possible worlds as shown earlier) may be deduced from the least action principle. Secondly, this principle does not operate on the level of the objective worlds but on the level of subjective points of view. In addition, the idea of the least, inherited from Maupertuis – the father of the principle of least action – is not correct as pointed out by Leibniz himself, since what counts here is not the minimum but only the fact that the derivative vanishes. A number of misunderstood Leibnizian notions are a direct consequence of such false and illegitimate correspondences. It is true that Leibniz paid a special attention to this kind of principles rooted in the work of Fermat on optics before being applied to mechanics, but the logical construction of the double infinite inclusive and exclusive notions associated with possible worlds and points of view should be distinguished from the principle of least action. Such an association made by a number of scholars does not only yield an extreme reduction of the Leibnizian methodology but it leads automatically to ambiguities and contradictions which are not to be located in the Leibnizian methodology itself but in the brain of the interpreter who proposes such fallacious and illegitimate analogies. Numerous scholar plans broke down because they defended Leibniz using erroneous and logically inconsistent arguments. ***In the absence of a real dynamical Leibnizian framework as the one developed in this work, it is impossible to understand the relevance of the Leibnizian assertions on physics and particularly on dynamics***. These misunderstandings correspond to the fact that Leibniz did not develop explicitly these ideas, giving only broad outlines on the subject matter.

Let us recall once more that ***the basic Leibnizian intuition lies in Aristotle's metaphysics where a net distinction is made between essence and its various modalities of existence***. Since physics does not respect such a distinction (in general and particularly in mechanics), all those who tried to defend Leibniz by use of the mechanistic paradigm have failed because this paradigm is too narrow to allow for a coherent statement of the Leibnizian principles.

Appendix I

Einstein's space-time arguments confronted with the Leibnizian dynamical ones.
The Einsteinian assertion, according to which his approach shows that Newton's mechanics remains locally valid, seems to be an interesting result as long as one deals with relativity theory starting with the usual kinematical framework before dealing with dynamics: if one recalls Lorentz transformations

$$r' = \gamma[r + Vt] \qquad t' = \gamma [t + Vr/c^2] \quad \text{with } \gamma = [1 - V^2/c^2]^{-1/2} \qquad \text{(I1)}$$

then one immediately shows that when $c \to \infty$ the Galilean transformations are recovered: ($r' = r + Vt$, $t' = t$). Here lies the major justification of Einstein's assertion. However, if one deals directly with dynamics following the line of thought proposed initially by Descartes and Huygens before being developed and emphasized by Leibniz, then one discovers a totally different situation. In particular, Einstein's assertion turns out to be much less significant since the parabolic Newtonian dynamical framework expressed usually through the following relation

$$E = p^2/2m + E_0 \qquad \text{(I2)}$$

may be interpreted as the local form of **any** even function, among which the hyperbolic Einsteinian one given by

$$E = mc^2 [1 + p^2/m^2c^2]^{1/2} \qquad mc^2 = E_0 \qquad \text{(I3)}$$

Thus, one does not need at all Einstein's structure anymore to find out the local validity of Newtonian dynamics. In Leibnizian terms, one may say that there exist an infinite number of possible dynamical worlds that reduce locally to the parabolic Newtonian one. **The idea of possible worlds is then directly related to dynamics and is not anymore a purely metaphysical entity with no relevance to physics**. On the other hand, if Einstein's dynamics is the only one that is admissible apart from the Newtonian one, then one should look at the basic arguments that single out this dynamics among the infinite number of other possibilities. Such a selection principle brings more intelligibility than the one provided by the usual historical procedure. In a sense, all those who look for rational arguments to single out the Einsteinian dynamics work partly in a Leibnizian paradigm. To work completely in such a paradigm, one should not content oneself with possible worlds, but should also account for an infinite multiplicity of points of view on each world. Before showing that the idea of a multiplicity of points of view on motion is also present implicitly in the dynamical world, let us note that the general reasoning linking global qualitative properties to local quantitative ones independently of any global quantitative specific structure (as done by Einstein), applies to the parabolic Newtonian Lagrangian: $L = ½ mv^2 + L_0$ with $L_0 = -E_0$. Again, if one considers any even function with respect to v, locally, one gets then a parabolic one.

In this regard, let us recall that according to Leibniz, a parabola may be considered as an ellipse one of its foci being cast to infinity. It is remarkable to note that if one replaces the parabolic Lagrangian by an elliptical one and applies to it the Lagrange-Hamilton formalism, then one obtains automatically Einstein's dynamics. Obviously, contrary to what is sometimes affirmed, Leibniz methodology is not intimately linked to the Lagrange-Hamilton formalism through the "least action principle". Thus, a simple application of the "least action principle" leading to Einstein's dynamics cannot be associated directly and faithfully with the Leibnizian methodology. If one carries Leibniz logic to its conclusion, one is led then to a richer structure than the one proposed by Einstein at two different levels: (i) multiplicity of exclusive possible worlds and (ii) inclusive points of view as shown below.

(i) **Possible worlds:** As already pointed out, one may recognize the local validity of Newton's dynamics without any need for Einstein's formulation, recalling that since a parabolic function is a local form of an infinite number of possible functions, then one finds here a natural dynamical justification of Leibniz emphasis on the idea of infinite multiplicity of possible worlds.

(ii) **Points of view:** As shown all along this work, Leibniz is the main defender of the importance of points of view on a given reality. We shall immediately show that this statement has also its natural place in the realm of dynamics. To see this, let us recall the following Newtonian parabolico-linear structure

$$E = ½\, mv^2 + E_0 \qquad p = mv \qquad \text{(I4)}$$

In spite of the conceptual difference between the **exclusive objective multiplicity** of possible worlds and the **inclusive subjective multiplicity** of points of view, the structural reasoning applied to possible worlds holds also for points of view. There exists however two main differences. Firstly, the reasoning is to be applied to both complementary even and odd functions as shown in the above two (parabolic and linear) relations. Secondly, this inclusive multiplicity should appear explicitly, since formal operations will link the different points of view, in an ordered manner leading to "inter-subjectivity" (as shown in Appendix G) which has no counterpart in the analytical method. Formally, one writes

$$E = mc^2[\, a + x_\mu^2/2 + \sum C_{\mu k}\, x_\mu^{2k} \,] \,,\; p = mc[\, x_\mu + \sum B_{\mu k}\, x_\mu^{2k-1}] \qquad x_\mu = v_\mu/c \quad k > 1 \qquad \text{(I5)}$$

At a first order approximation (none of the coefficients $C_{\mu k}$ and $B_{\mu k}$ intervenes) the multiplicity of points of view vanishes. If such an explicit multiplicity does not appear in the Newtonian framework, it is because locally the different points of view become fused with each other and then, impossible to be detected. In other words, **the "one" and the "many" cannot be distinguishable locally but only globally.**

We see here that contrary to the reasoning on kinematics ($v = dr/dt$), the reasoning on dynamics leads to different perspectives which are totally compatible with the Leibnizian propositions. These appear to be illogical and contradictory if applied to kinematics. Here lies one of the main reasons for which Leibniz was much denigrated by those who rooted their arguments on purely kinematical arguments directly linked to the Lagrange-Hamilton formalism. With such a restriction there is no way to understand the Leibnizian system combining **exclusive** and **inclusive** multiplicities related respectively to **possible worlds** (from which one should select the best) and to different **points of view** on each world.

Let us finally emphasize that this multiplicity of points of view is not a question of choice but a necessity as shown in Appendix F. This becomes clear if one recalls that the velocity concept is bounded and corresponds to an asymptotical behaviour, where no measurement is possible anymore at the level of the asymptotical behaviour as discussed elsewhere in this work.

Appendix J

The "usual" and "emergent" rationalities linked to a work due to Taylor and Wheeler.
The importance of the dynamical concepts such as energy and impulse lies in the fact that these concepts play a central role in the basic framework of physics, independent of any one of the four physical interactions (gravitational, electromagnetic, weak or strong interaction). As pointed out by Wigner, followed by Lévy-Leblond and many other scientists and epistemologists, the laws governing this somehow foundational pre-physical substrate which also intervene through different couplings at the level of the various physical interactions present a privileged statue. These are called **"super-laws"** distinguished thus from the more particular laws governing the specific interactions. The "usual rationality" and the "emergent one" as well as some other considerations developed in this appendix, concern precisely the basic framework on which the physical theory is founded. In conventional physics and particularly in the so-called space-time physics inherited from Newton, one distinguishes between three basic entities: space-time-matter or length-duration-mass. The different other concepts such as velocity, force, energy, impulse etc. **derive** from these three basic elements. However, with the advent of modern 20th century physics (relativistic and quantum revolutions) the concepts associated with space and time have lost their privilege, particularly because of the existence of some physical phenomena that cannot be dealt with in the space-time framework. Such phenomena that do not enter into the mould of space-time and its associated rationality led some scientists to look for other possible frameworks among which the "emergent rationality" developed on a dynamical ground (inverting thus the usual procedure). It is not the space-time couple (x, t) which is privileged but the impulse-energy couple (p, E) which is considered to be primary. In this Appendix, we recall the two rationalities, each of which constituting one point of view, before examining their link to a third standpoint evoked by Taylor and Wheeler which does not constitute an autonomous framework comparable to the "usual and emergent rationalities". Finally, it is shown that Leibniz formulation may also be linked directly to Taylor-Wheeler standpoint in addition to its inclusion of the different above-mentioned rationalities.

Usual rationality.
The three conservation laws of Newtonian dynamics: $\frac{1}{2} m v^2$, $m v$ and m are linked together through successive derivations. Instead of proposing a direct justification of this fact through the relativity principle, Newtonians among which Lagrange and Hamilton proposed a totally different articulation that constitutes the "usual rationality" of physics. This rationality runs as follows:
After the introduction of the concepts of **space** and **time** to define the velocity, one enters into the realm of dynamics by assuming the existence of a positive constant, the **mass** combined with the velocity to obtain what is now known as the Lagrangian from which the two quantities corresponding to conservation laws of energy and impulse are obtained as follows:

$$p = dL/dv \qquad E = v\, dL/dv - L \qquad \text{(J1)}$$

These are deduced from the well-known "principle of least action".

The knowledge of the Lagrangian fixes the associated dynamics so that on setting

$$L = \frac{1}{2} m v^2 \text{ (Newton)} \qquad L = - mc^2 [1 - v^2/c^2]^{1/2} \text{ (Einstein)} \tag{J2}$$

one deduces the Newtonian and Einsteinian expressions of impulse and energy as follows:

$$p = mv \qquad E = \frac{1}{2} mv^2 \text{ (Newton)} \tag{J3}$$

and

$$p = mv/ [1 - v^2/c^2]^{1/2} \qquad E = mc^2 /[1 - v^2/c^2]^{1/2} \text{ (Einstein)} \tag{J4}$$

Emergent rationality.
The existence of the "emergent rationality" appears clearly in the work of C. Comte and particularly in Ref.[16], where he shows that the ideas of Langevin on dynamics deserve to be actualized by use of group theoretical methods. They seem to constitute a better foundation for relativity than those proposed by the first discoverers starting with Lorentz, Poincaré and mainly Einstein. The author also evokes in Ref.[16], J.M. Lévy-Leblond according to which " *it will be of a great usefulness to dispose of a direct relativistic dynamics, from which the properties of space-time would be subsequently deduced"* (my translation Ref.[16] p.107). C. Comte concludes by presenting Langevin's approach as a general program of geometrization of physics, including the quantum domain. In so far as the "rapidity" concept is concerned, the basic constraint encountered in this work has also been developed in a previous work [19] recalled by Jammer [18] {p. 47, Eqs.(2.17)-(2.18)} and reproduced here as follows

$$f(U + V) + f(U - V) = 2f(U) f(V) \tag{J5}$$

where the "rapidity" is called "pseudovelocity" in this work.

This equation is the same as the one given by Comte in Eq.(9) of Ref.[16], but the basic postulates are not exactly the same. In particular, C. Comte obtains (J5) by use of the principle of relativity associated with the isotropy requirement. This equation may be put into a differential form. Differentiating (J5) twice with respect to V and putting V = 0 yields the following second order differential equation

$$f''(0) f(U) = f''(U) \tag{J6}$$

from which one deduces an even (isotropy criterion) function and its derivative, an odd one as follows:

$$f(U) = \cosh U \qquad f'(U) = \sinh U \tag{J7}$$

These functions are intimately linked to energy and impulse, respectively. They are obtained after imposing appropriate limit conditions or equivalently the isotropy requirement. More precisely, the counterparts of Eqs.(J4) associated with the “usual rationality” correspond in the “emergent rationality” to

$$p = mc \sinh w/c \qquad E = mc^2 \cosh w/c \qquad (J8)$$

where we have set

$$w/c = U \quad f(U) = E/mc^2 = M/m \quad \text{(different notation: } f(U) = m/m_0 \text{ in Refs.[18,19])} \qquad (J9)$$

The function f(U) corresponds to the ratio between “relativistic mass” and “invariant mass”(m/m_0) in Refs.[18,19] and to the ratio between energies in Ref.[16] (E/mc^2). One may refer to other works and other methods associated with the “emergent rationality” [11,14] particularly those proposed by J.M. Lévy-Leblond and J.P. Provost. In spite of the formal analogies between the different approaches, one should recognize that C. Comte is the author who examined this “emergent rationality” on general grounds, proposing different complementary approaches [10,15,16]. Let us emphasize that, **whatever the approach and the authors; there is one major property on which the “emergent rationality” is based. It concerns the additive character of the composition law associated with motion (rapidity).** Its relation to the velocity concept is given by the following relation:

$$v = c \tanh w/c \qquad (J10)$$

As to the composition of motion, it corresponds to

$$w' = w + W \quad \text{(for rapidity)} \qquad v' = v * V = [v + V]/[1 + vV/c^2] \quad \text{(for velocity)} \qquad (J11)$$

Let us also note that although the main works performed by C. Comte and J.M. Lévy-Leblond on the subject matter focus on these two complementary points of view there is a preference to the rapidity concept, especially in the work of C. Comte who shows the existence of a dynamical solution for which the first Hamilton canonical equation for free particles (v = dE/dp) is not verified as explained in page 105 of Ref.[16]. In this work C. Comte asserts that this “rational framework” may be comparable in its importance to that of Lagrange and Hamilton, (applicable in different contexts including quantum mechanics) provided one works in the appropriate mathematical framework of group theory. In Ref.[13] J.M. Lévy-Leblond goes beyond the two points of view associated with “usual (velocity) and emergent (rapidity) rationalities” by evoking a third point of view that he called “celerity”. Strictly speaking, this point of view is not completely new since it is encountered in the usual approach of relativity theory under the name of “proper velocity”. One should however recognize that the new feature (emphasized by J.M. Lévy-Leblond) is the consideration of celerity as one point of view on motion among others which is not the usual interpretation in the conventional approach.

Link to propositions provided by Taylor and Wheeler.

Taylor and Wheeler seem to be the authors who favoured this third point of view in such a way that it may be considered as fundamental too. In page 159 of Ref.[12], they explain that, since the most useful quantities in physics are the invariant ones, for they keep the same value in all reference frames, it is then important to favour them in the definition of motion. Thus, it is more satisfactory to define impulse in relation to proper time τ instead of the improper one t (the fourth component of the space-time vector). The first one is measured in a co-moving frame of reference while the second is measured in a fixed (laboratory) frame. Thus, one writes

$$p = m\, dr/d\tau \quad \text{instead of} \quad p = M\, dr/dt, \quad \text{with } M = m /[\, 1 - 1/c^2(dr/dt)^2]^{1/2} \qquad \text{(J12)}$$

(as done in some old books where M corresponds to “relativistic” or “ modified” mass). This mass (equivalent to energy $E = Mc^2$) is not invariant since its value differs in the passage from one frame of reference to another. According to Taylor and Wheeler, in favouring the invariant couple (τ, m) with respect to the non invariant one (t, M) one gains in intelligibility (invariance) as well as in simplicity (impulse is proportional to motion exactly as for Newtonian dynamics). If such a criterion is to be used in the present Leibnizian framework, one should be conscious of the fact that the kinematical arguments proposed by Taylor and Wheeler cannot be considered in the present Leibnizian approach, since in such an approach, the kinematical relations such as Lorentz transformations and their invariance should not be taken into consideration for Leibniz’s main idea was to deal directly with dynamical arguments. This is possible thanks to the degrees of freedom provided by the different ways to deal with conservation laws. Instead of being confined into a too narrow framework from the start, with a strong constraint such as the one provided by the velocity concept, the Leibnizian **“principle of plenitude”** consists in taking the widest class of solutions (multiplicity of points of view) compatible with the properties of conservation (developed in the first part of this work). Thus, among the, a priori, unlimited multiplicity of points of view on motion related to impulse through odd functions as follows

$$p = g^{\mu}(v_{\mu}) = g^{\mu}(-v_{\mu}) \qquad \text{(J13)}$$

it is natural to consider one of those points of view to be proportional ($\mu = p$) to impulse writing thus

$$p = g^{p}(v_{p}) = mv_{p} \qquad \text{(J14)}$$

Physically, the proportionality relation defines, here, the mass concept yet undefined since the principle of dynamical relativity deals only with **conserved entities** which **vary in the passage from one frame of reference to another,** contrary to the **mass** concept which **remains invariant**. It should be also noted that, what is usually called “relativistic” or “modified” mass plays a central role in the construction of the present approach, but it disappears afterwards like scaffolding that one removes after the building has been achieved. To see this, let us recall that, in the present approach, the process of derivation plays the role of a generator of conservation laws and that one should impose a constraint on the second derivative to limit the number of laws to two. The second derivative with respect to motion, having the dimension of a mass (depending on motion in general), shows that this sort of mass corresponds precisely to what is known as the “relativistic” or “modified” mass. However, unlike the usual old interpretation leading to a certain misconception of relativity theory as emphasized by Taylor and Wheeler, in the present approach this entity allows the determination of the solution, since it corresponds to the constraint imposed on the structure to satisfy the relativity requirement. It plays then **an essential role in the**

construction and leads directly to the famous relation $M = E/c^2$, except that here M is defined as a second order μ- derivative (extended derivative) as follows:

$$M = d_\mu{}^2E/dv_\mu{}^2 = D_\mu\, d/dv_\mu\, [D_\mu\, dE/dv_\mu] = E/c^2 \qquad \text{(J15)}$$

where the Greek index μ accounts for the unlimited multiplicity governed by a recurrent series in the general case as shown elsewhere in this work. As to impulse, it derives from energy through

$$p = d_\mu E/dv_\mu = D_\mu\, dE/dv_\mu \qquad \text{(J16)}$$

The above two equations are none other than Eqs.(5, a) and (5, b) fully justified in the main text.

Taylor and Wheeler approach as a particular perspective of the Leibnizian methodology.
The definition of the invariant mass as done above through the proportionality relation in (J14) imposes such a strong constraint that it leads automatically to the structure of relativity theory proposed by Taylor and Wheeler. One should nevertheless precise that these authors do not pretend to construct a new rationality as done by C. Comte for the “emergent rationality”, widely developed in high energy physics. Taylor and Wheeler recognize that the most natural point of view on motion is given by the rapidity called in Ref.[12] by “velocity parameter”. Their argument runs as follows: they begin by establishing an analogy between hyperbolic functions and trigonometric ones before explaining that the hyperbolic angle which corresponds to the “rapidity” that they call the “velocity parameter” is the most natural one to deal with, in dynamics. Once the analogy is accepted, the “velocity” concept corresponding to the **slope** while the rapidity corresponding to the **angle**, these authors explain that if the measure of motion through rapidity is more convenient, it is because “ ***the angles are additive while the slopes are not*** ” (see page 65 of Ref.[12]). After these considerations, let us go back to the concept of “proper velocity” ($u = dr/d\tau$) also called “celerity” by Lévy-Leblond and to the resolution of (J15) and (J16) subject to the constraint (J14).To this end, one substitutes the fixed point of view given in (J14) into (J15)-(J16) and deduces

$$D_p\, m = E/c^2 \;, \quad (E/mc^2)d/dv_p\, [(E/mc^2)\, dE/dv_p] = E/c^2 \qquad \text{(J17)}$$

whose solution corresponds to

$$E^2 - m^2c^2v_p{}^2 = A + B\, v_p \qquad \text{(J18)}$$

Where A and B are integration constants. This solution reduces to

$$E^2 - c^2p^2 = m^2c^4 \qquad p = m\,u \qquad u = v_p \qquad \text{(J19)}$$

if the following limit conditions are chosen ($p = 0$, $u = 0$, $E = mc^2$). Thus, one is finally led to the following two conservation expressions:

$$p = mu \qquad E = mc^2\, [1 + u^2/c^2]^{1/2} \qquad \text{(J20)}$$

One easily notices that whatever the point of view, the three different relations (J20), (J8) and (J4) lead locally to the following parabolico-linear structure

$$P = m\,u \quad E = \tfrac{1}{2}\,mu^2 + mc^2 \quad \text{with} \quad u = w = v \tag{J21}$$

We have constrained the Leibnizian qualitative differential system of equations (J15) in this way in order to show that in spite of the analogy that occurs between the Leibnizian second order differential equation and that of the "emergent rationality", there is however a substantial difference. Unlike the "emergent rationality" which is constrained **uniquely** by the rapidity parameter, the Leibnizian qualitative second order differential equation may be subject to an infinite number of parameterizations as shown in the main text. In particular, the deviator which is not yet defined, can be adapted to different sorts of constraints, among which the one proposed by Taylor and Wheeler, corresponding neither to the "usual rationality" nor to the "emergent one". This shows the importance of having a general qualitative principle of relativity unconstrained by one specific criterion or another. Such criteria appear at a later stage and may be chosen according to the particular problem one is interested in. In particular, on introducing a "principle of order" (plenitude) one may use a "simplicity criterion" that differs significantly from the "simplicity criteria" evoked when dealing with one model or another. The difference lies in the fact that in the Leibnizian approach the "simplicity criterion" operates at the "inter-subjective" level (through the ordering of the points of view with respect to each other) while usually it operates on a specific point of view associated with some remarkable property (proportionality relation as in (J20), additive parameter as for the "emergent rationality, etc.). The Leibnizian approach, being a **theory** and not a **model**, it is capable to generate simultaneously, different remarkable properties instead of postulating them in association with a specific methodology.
Let us finally notice that among the different parameterizations, the one selected in (J17) [corresponding to the point of view underlined by Taylor and Wheeler], is the simplest one: the energy cancels out from both sides of the equality leading to $d/dv_p\,[(E/mc^2)\,dE/dv] = m$ whose integration is immediate.

Comment on the Leibnizian qualitative dynamical thought confused with topology.
Let us finally notice that the existence of such a qualitative principle of relativity capable to adapt itself to a multiplicity of points of view is a typically Leibnizian idea associated with the realm of dynamics and not with that of topology as asserted by a number of authors. Indeed, a number of scholars interpret the emphasis of Leibniz on qualitative methods as a proof of his interest in topology which is precisely based on qualitative arguments. However, as shown in Ref.[4] page 160, Leibniz's qualitative thinking is not to be associated with qualitative geometry called topology. According to L. Bouquiaux [4] the ambition of the "analysis situs" was to reduce all geometrical propositions to propositions bearing on distances between points. Obviously, this ambition is directly related to Leibniz rejection of absolute space where only the relative positions with respect to each other really count. Such a Leibnizian treatment of distance is not a qualitative notion. Leibniz did not question in his "analysis situs" invariant figures under continuous deformations, nowadays associated with so-called topological deformations. B. Mates too [30] (page 240), proclaims that one should avoid the assimilation of the Leibnizian "analysis situs" to topology, this misconception being due to Euler who seems to have been at the origin of this

confusion. It is Euler who attributed the paternity of topology to Leibniz since he is the first author who used the term “analysis situs” to indicate what is nowadays known as topology. In addition, it should be emphasized that the Leibnizian “relational” approach of nature is not to be reduced to the fact that what counts is not the idea of a point in space or space-time but a point in relation to all other points. Let us also recall that, in another respect, it is also Euler who convinced Kant to adopt the views developed by Newton rather than those developed by Leibniz in so far as natural phenomena are concerned as shown in Ref.[23]. Euler is certainly a great mathematician but some of his philosophical analyses seem to breed more darkness than it dispels.

Appendix K

From the Newtonian quantitative to the Leibnizian qualitative framework and development of the dynamical relativity principle in its general form.

First part: Passage from the Newtonian to the Leibnizian framework.
When considering the following Newtonian dynamical equations

$$E = \tfrac{1}{2} mv^2 \quad p = mv \tag{K1}$$

used in the study of elastic collisions, one immediately notices that not only impulse p derives from energy E as follows

$$p = dE/dv \tag{K2}$$

but also the concept of mass corresponds to a first derivative of impulse or second derivative of energy $M = dp/dv = d^2E/dv^2 = m$. These remarkable properties are usually considered to be purely accidental by most physicists aware of two major facts. Firstly, impulse does not derive from energy but from the Lagrangian and secondly, the Newtonian framework is local, so that two different entities having the same dimension will coincide locally since there is not enough place for any variety in such a local framework. As long as one associates the parameter v with the usual velocity concept, one should recognize that the Lagrange-Hamilton formalism imposes itself. However, if one adopts other points of view on motion, new interpretations become possible. Leibniz invites us to focus the attention on the **necessity of conservation laws** and on the possibility of dealing with such laws through different manners, each of them corresponding to one point of view. Being the father of differential calculus and having lived long before the advent of the Lagrange-Hamilton formalism, Leibniz paid much attention on the **passage from one entity to the other through differentiation and integration** as shown in Refs.[2,4,9]. In particular, looking for a complete or an integral rationality (according to the tradition inherited from Descartes “grand rationalisme”) with no recourse to any experimental justification, Leibniz remarked that on pursuing the different derivations, then, no additional conserved entities that depend on motion are obtained. Thus, the second order differential equation, corresponding to the Newtonian mass concept, plays the role of a constraint that forbids the obtainment of more than two conservation laws required in the study of elastic collisions. Adopting such a link between the different entities, leading to a particular interpretation, one gets the possibility of deducing both energy and impulse by simply imposing the following constraint:

$$d^2E/dv^2 = c^{te} \text{ or more explicitly } dp/dv = c^{te}\ , \quad p = dE/dv \qquad (K3)$$

with the limit conditions:

$$v = 0,\ p = 0 \text{ and } E = 0 \qquad (K4)$$

From the reign of quantity to that of quality.

This interpretation that differs substantially from that of Lagrange and Hamilton may be directly extended in two directions, replacing the derivation operator d/dv by a qualitative operator $O = d_\mu/dv_\mu$ and the constant constraint $M = m{:}c^{te}$ by a more general one $M = C(E,p)$ compatible with the necessities provided by the conservation requirement. Thus one is left with the formal qualitative and undetermined expressions

$$\mathbf{M = O^2(E) = C(E, p) \text{ or more explicitly } O(p) = C(E, p)\ , \quad p = O(E),} \qquad \mathbf{(K5)}$$

This constitutes a generalization of the quantitative and well-determined dynamics given in (K3). Obviously, it should be emphasized that the internal logic of the Leibnizian methodology is not necessarily linked to the Newtonian framework through Eqs.(K1) as shown in the main text. However, on starting with the Newtonian framework, one shows that the Lagrange-Hamilton formalism does not impose itself but there are different possible interpretations of things among which the (historically neglected) Leibnizian interpretation which turns out to be fruitful as shown all along this work. Its fruitfulness is a direct consequence of two basic Leibnizian considerations: ***firstly, Leibniz invites us to focus the attention on the necessary requirements associated with the conservation properties in relation with the relativity principle. Secondly, he adopts the "principle of plenitude" asserting that one should consider all the degrees of freedom compatible with the above necessary requirements.***

Comments on the local Lagrange-Hamilton formalism and on Leibniz's methodology

Although the Leibnizian interpretation (as well as its natural extension from the realm of quantity to that of quality), differs from the Lagrange-Hamilton formalism, it may nevertheless be linked to it. Let us firstly note that, according to the Lagrange-Hamilton formalism, energy is not deduced by imposing a constraint on the second order derivative, but it is deduced from the Lagrangian as follows:

$$L = \tfrac{1}{2} mv^2 \Rightarrow p = dL/dv = mv \text{ and } E = vdL/dv - L = mv^2 - \tfrac{1}{2} mv^2 = \tfrac{1}{2} mv^2 = p^2/2m \qquad (K6)$$

This clearly shows that **whenever the Lagrangian L differs from ½ mv^2**, the **energy** E (also called the Hamiltonian) **differs also from this parabolic form**. In particular, in Einstein's dynamics, the Lagrangian is elliptic $[(L/mc^2)^2 + (v/c)^2 = 1]$ and the Hamiltonian is hyperbolic $[(H/mc^2)^2 - (p/mc)^2 = 1]$. Leibniz, much interested in conics and in their analytical forms knew very well that the process of localization transforms both the elliptic and hyperbolic forms into a parabolic one. This explains why Leibniz critics of 17th century mechanics were based on a firm ground. This also explains why the natural extension of a parabolic form leads automatically to the class of even functions whose localization leads automatically to a parabolic form. Hence, **the idea of possible worlds or possible dynamics is not only rooted in Leibniz metaphysics, as usually believed, but also in his mathematical and dynamical discoveries.** Let us also emphasize that Leibniz asserted at different occasions that his qualitative metaphysics is intimately related to his quantitative mathematical findings and reciprocally so that the passage from the realm of quality to that of quantity was permanent in Leibniz methodology.

If one reduces dynamics to the rational framework provided by Lagrange and Hamilton, then the above qualitative extension of the Newtonian framework looses its meaning and becomes impossible to grasp. But if one recognizes that other possible rationalities may exist, then **the above qualitative extension leads naturally to a Leibnizian framework, through the distinction between two kinds of multiplicities: exclusive and inclusive ones, associated respectively with possible worlds and points of view on each world**. More precisely, the multiplicity of possible worlds corresponds to the **different sort of constraints** C(E, p) and the multiplicity of points of view on each world corresponds to the **multiple character of the operator** O explicitly given by

$$O = d_\mu/dv_\mu = D_\mu\, d/dv_\mu = D_\beta\, d/dv_\beta = d_\beta/dv_\beta \qquad \text{(K7)}$$

The Greek indices indicate the multiplicity of points of view (specified in Appendices A and B). Here the attention is focussed on the fact that **the Newtonian structure constitutes a germ, from which one is directly led to a dynamics of a Leibnizian type.**

In spite of the difference between the Leibnizian procedure and the Lagrange-Hamilton one, it should be recalled that the two procedures deal with a scalar concept (energy) so that one may establish some link between both procedures. In particular, the elimination of the infinite multiplicity of points of view in favour of a relation between energy and impulse can be obtained as follows:

$$O(E) / O(p) = dE/dp = p/ C(E, p) \equiv v$$

One recognizes the first Hamilton canonical equation of a free particle, if one associates v with the velocity concept ($v = dE/dp$). This allows saying that the above mechanism of compensation (**eliminating motion in favour of impulse**) constitutes the **counterpart of the Legendre transformation** whose goal is precisely to express energy in terms of impulse instead of velocity.

The fundamental difference is that, instead of dealing only with one parameter associated with motion (velocity), one deals here with an, a priori, infinite number of parameters (v_μ multiple points of view on motion). Another difference is that the structure of the Leibnizian dynamical relativity principle suggests to deal with the inverse of v and not with v itself ($p' = dp/dE$ and not dE/dp) as shown in Eqs.(4, a)-(4, c) developed only in the particular case corresponding to $\gamma = 0$. The second part of this Appendix is devoted to the study of the general solution $\gamma \neq 0$.

Second part: development of the most general expression associated with the dynamical relativity principle.

The application of the dynamical relativity principle given in (K5) and (K7) (or in its trans-subjective version) leads to

$$M = O^2(E) = d_\mu{}^2E/dv_\mu{}^2 = pdp/dE = C(E, p) \qquad \text{(K8)}$$

where the different points of view are eliminated in favour of the conserved entities (trans-subjective mechanism). On assuming a separation of variables, Eq.(K8) transforms to

$$pp' = C(E, p) = f(E) + g(p) \qquad p' = dp/dE \qquad \text{(K9)}$$

On deriving (K9) twice with respect to E one deduces

$$pp''' + 3\, p'p'' = f\,''(E) + g''(p)\, p'^2 + g'(p)\, p'' \qquad \text{(K10)}$$

The interest of such a formal expression lies in the fact that the conservation laws are subject to the following constraint

$$f\,''(E) = 0 \text{ and } g''(p) = 0 \qquad \text{(K11)}$$

[as shown in the first part of this work through Eq.(4, a)]. Thus, Eq.(K10) reduces to

$$pp''' + 3\, p'p'' = g'(p)\, p'' = \gamma\, p'', \quad \gamma = c^{te} \qquad \text{(K12)}$$

or equivalently, to

$$p'''/p'' + 3\, p'/p = g'(p)/p = \gamma/p \qquad \text{(K13)}$$

The elimination of $\gamma = c^{te}$ by deriving (K13) multiplied by p, leads to a fourth order differential equation as follows:

$$\mathbf{[p\, p'''/p'' + 3\, p']' = 0 \Leftrightarrow p'p''p''' + p\, p''p'''' - pp'''^2 + 3\, p''^3 = 0, \quad p' = dp/dE} \qquad \textbf{(K14)}$$

This last equation is the most general differential equation associated with the dynamical relativity principle. The only entities that occur in such an equation correspond to conserved quantities, namely energy and impulse (E, p). The four constants associated with this fourth differential equation will permit to define a broken parity (anisotropy) constant γ, an invariant mass m, a coupling constant c (velocity of light or upper limit velocity) and an upper limit energy E_M.

Let us note that when the constant γ vanishes so that one recovers the isotropy requirement, then not only $(K14)_1$ vanishes but also its primitive $p\, p'''/p'' + 3\, p' = 0$ as shown in (K13). This third order differential equation may be integrated easily leading to

$$p^3\, p'' = A \qquad \text{(K15)}$$

where A is a constant of integration. The situation is more complicated when γ does not vanish. In this case, one gets the following integro-differential equation

$$p^{3-\gamma/p'}\, p'' = A \exp[\gamma \int \{L|p|\, p''/p'^2\}dE] \qquad (L|p|)' = p'/p \qquad \text{(K16)}$$

It would have been more rigorous if we had expressed the couple (p, E) in a non-dimensional framework ($p^* = p/p_0$, $E^* = E/E_0$) so that $(L|p^*|)$ replaces $(L|p|)$ but this complicates the expressions uselessly.

The differential Eq.(K16) constitutes the counterpart of Eq.(K15) except that the isotropy requirement is not imposed (invariance under the symmetry $p \to -p \Rightarrow \gamma = 0$). The lack of such a symmetry deforms both the exponent 3 and the constant of integration A: the first is to be replaced by $3 - \gamma/p'$ and the second by $A \exp[\gamma \int \{L|p|\, p''/p'^2\}dE]$. Obviously, in the absence of such a constraint the deformation is absorbed. Thus, (K16) reduces to (K15) when $\gamma = 0$. A simple derivation of Eq.(K16) shows that it is compatible with (K12).

Let us conclude this appendix by noting that it is mathematically natural to set $\gamma = 0$, since in this case, the mathematics is much simpler than with $\gamma \neq 0$. The more the hypotheses are weak, the more the mathematics is rich. One may say that isotropy is not a physical necessity but its absence breaks a certain symmetry leading to additional difficulties. This fact exhibits the **importance of the principle of simplicity in the development of physical theories,** at least in a first step, so that one may cope with the difficulties progressively.

Appendix L

Descartes extension through a regularization procedure: a fruitful one.

In order to show that Leibniz's proposition concerning the fact that Descartes dynamics may be valid at some scale, one should proceed as follows. One starts by deriving (three times with respect to E), Descartes formula $p^2 = a_1 E^2$ [given at the end of the first part of this work through Eq.(25)], showing that it is compatible with the relativity principle given through Eq.(4, c). Then one integrates the obtained differential equation and benefits from the process of integration in order to obtain a regular solution of the following form

$$p^2 = a_1 E^2 + \kappa f(E) \qquad \text{(L1)}$$

Obviously, **f(E) should vanish when $E \to E_M$ since Descartes dynamics is assumed to be valid only at this scale** (in the vicinity of E_M). Thus, one writes

$$f(E) \text{ such that } f(E_M) = 0 \qquad \text{(L2)}$$

or equivalently,

$f(E) = g(E - E_M)$ such that $g(0) = 0$ (L3)

The process of derivation and integration associated with the trans-subjective version of the dynamical relativity principle expressed through Eq.(4, c) imposes the following constraint on $f(E)$ and $g(E - E_M)$.

$f'''(E) = 0 \Rightarrow f(E) = ½ AE^2 + BE + C$ (L4)

$g'''(E - E_M) = 0 \Rightarrow g(E - E_M) = ½ F(E - E_M)^2 + G(E - E_M) + H$ (L5)

where A, B, C and F, G, H are integration constants associated with the functions f and g respectively. On account of (L2) and (L3) one may determine the two constants H and C

$H = 0 \qquad C = - ½ A E_M^2 - BE_M$ (L6)

As to the initial condition requirement

$p = 0 \quad E = E_0$ (L7)

that replaces the Cartesian one which is not valid at the origin but only for $E \rightarrow E_M$, the latter transforms Eq.(L1) as follows

$p^2 = a_1E^2 + \kappa f(E) = a_1 [E^2 - E_0^2 f(E)/f(E_0)]$, $\qquad \kappa = - a_1E_0^2/f(E_0)$ (L8)

The combination of the different equations given in (L4)-(L6) leads to

$f(E) = g(E - E_M) = ½ F(E - E_M)^2 + G(E - E_M) = ½ A(E^2 - E_M^2) + B(E - E_M)$ (L9)

with the following relations between the constants of integration

$A = F \qquad B = G - FE_M$ (L10)

These different coupled solutions suggest looking for three uncoupled ones proportional to: $(E - E_M)^2$, $(E^2 - E_M^2)$ and $(E - E_M)$ respectively. This may be written in a compact form following the line of thought associated with Leibniz's methodology of discrete possible worlds already dealt with in the first part of this work. Thus one is finally led to

$p^2 = a_1 \{ E^2 - E_{0kl}^2[1 - (E/E_M)^k]^l \}$ (L11)

where we have set

$E_{0kl} = E_0 / [1 - (E_0/E_M)^k]^{l/2}$ (L12)

with

$(k, l) = \{(1, 1) ; (1, 2) ; (2, 1)\}$ (L13)

The first couple corresponds to F = A = 0, the second to G = 0 and the third to B = 0.
These solutions are highly significant since they do not only generalize Descartes dynamics and reduce to it when $E \rightarrow E_M$ but they also generalize Einstein's dynamics reducing to it when $E_M \rightarrow \infty$ or equivalently when energy is negligible with respect to E_M. In this last case, one gets

$$p^2 = [E^2 - E_0^2]/c^2 \quad \Leftrightarrow \quad E^2 - c^2 p^2 = E_0^2 \tag{L14}$$

where we have set

$$a_1 c^2 = 1 \tag{L15}$$

identifying thus the structural constant a_1 with the inverse of the square of light velocity. Before the end of this Appendix, let us notice that the solutions associated with the two last couples given in (L11) are the same as those constructed through different procedures in the framework of so-called "doubly special relativity" [7, 8]. Eqs.(L11) may also be expressed in such a way that they can be related to the usual fundamental equation of Einstein's dynamics: $E^2 - c^2p^2 = E_0^2$. To this end, one gets the following form

$$E_{kl}^2 - c^2 p_{kl}^2 = E_{0kl}^2 \quad E_{kl} = E / [1 - (E/E_M)^k]^{l/2} \quad p_{kl} = p / [1 - (E/E_M)^k]^{l/2} \tag{L16}$$

or explicitly,

$$E_{12}^2 - c^2 p_{12}^2 = E_{012}^2 \quad E_{12} = E / [\, 1 - (E/E_M)] \quad p_{12} = p / [\, 1 - (E/E_M)] \tag{L17}$$

$$E_{21}^2 - c^2 p_{21}^2 = E_{021}^2 \quad E_{21} = E / [\, 1 - (E/E_M)^2]^{l/2} \quad p_{21} = p / [\, 1 - (E/E_M)^2]^{1/2} \tag{L18}$$

$$E_{11}^2 - c^2 p_{11}^2 = E_{011}^2 \quad E_{11} = E / [\, 1 - (E/E_M)]^{l/2} \quad p_{11} = p / [\, 1 - (E/E_M)]^{1/2} \tag{L19}$$

When $E_M \rightarrow \infty$ one recovers Einstein's dynamics given in Eq.(14). The system associated with Eqs.(L17) corresponds to Maguejo and Smolin model [7], while the one associated with (L18) corresponds to Hinterleitner model [8].

It is remarkable to see that Leibniz's thought on dynamics is fruitful to modern physics. This is a direct consequence of the generality of his approach and his emphasis on the necessary and basic properties associated with conservation laws as well as with the dynamical relativity principle.

Introduction of a multiplicity of points of view in "doubly special relativity".
Noting that the hyperbolic structure associated with E and p is similar to the one associated with doubly special relativity provided one makes the following correspondences: $p \rightarrow p_{kl}$ and $E \rightarrow E_{kl}$ then, if one replaces p and E by p_{kl} and E_{kl} defined in Eqs. (L16)-(L19) one gets (see Eqs.(15)-(16) in the main text).

$$X_{kl} = cp_{kl} / E_{0kl} = x_{1kl} = \sinh x_{2kl} = \tan x_{3kl} = x_{4kl} / [1 - x_{4kl}^2]^{1/2} \quad \text{(L20)}$$

$$Y_{kl} = E_{kl} / E_{0kl} = [1 + x_{1kl}^2]^{1/2} = \cosh x_{2kl} = \sec x_{3kl} = 1 / [1 - x_{4kl}^2]^{1/2} \quad \text{(L21)}$$

where one deduces from (L12) and (L16) the following relations

$$E_{kl} / E_{0kl} = (E/E_0) [1 - (E_0/E_M)^k]^{1/2} / [1 - (E/E_M)^k]^{1/2} \quad \text{(L22)}$$

$$cp_{kl} / E_{0kl} = (cp/E_0) [1 - (E_0/E_M)^k]^{1/2} / [1 - (E/E_M)^k]^{1/2} \quad \text{(L23)}$$

Appendix M

Subjective version of the dynamical relativity principle applied to space-time physics and link to Lagrange-Hamilton Formalism and Taylor-Wheeler Approach.

The methodology developed in this work may be adapted to space-time physics associated either with the usual method of rational or analytic formulation due to Lagrange and Hamilton, or with the method proposed by Taylor and Wheeler, underlying the importance of invariance properties in dealing with physics. It should be recalled that both procedures are associated with the structure of space-time, since the first of these is based on the velocity concept, while the second deals with the celerity (proper velocity) defined through proper time (which is an invariant measure). The present procedure consists in replacing one of the postulates in each case, by the subjective version of the dynamical relativity principle. More precisely, instead of postulating two different methods each of them leading to one specific point of view on motion, these two different methods may be replaced by the application of the subjective version of the dynamical relativity principle. The method associated with the velocity is the one provided by the well-known Lagrange-Hamilton formalism. As to the one developed by Taylor and Wheeler, it focuses the attention on mass and proper time, both being invariant in all inertial systems as it will be shown at the end of this Appendix. We first show how the subjective version of the principle of dynamical relativity applies in connection with kinematical relativity, leading to the expressions of energy and impulse (E, p) in terms of the velocity and proper velocity (v, u). Then we recall briefly the ways borrowed by the Lagrange-Hamilton formalism and Taylor-Wheeler method.

Application of the subjective version of the principle of dynamical relativity to space-time physics.

Starting with Lorentz transformations with c = 1 (natural system of units):

$$r' = \gamma [r + Vt] \qquad t' = \gamma [t + Vr] \qquad \gamma = [1 - V^2]^{-1/2} \quad \text{(M1)}$$

as usually done in space-time physics, one is left with the following composition law associated with the velocity

$$v' = [v + V]/[1 + vV] , \quad v = dr/dt \quad v' = dr'/dt' \quad \text{(M2)}$$

Since the velocity v is related to proper velocity ($u = dr/d\tau$, $dt^2 - dr^2 = dt'^2 - dr'^2 = d\tau^2$), by $v = u/ [1 + u^2]^{1/2} \Leftrightarrow u = v/ [1 - v^2]^{1/2}$ where the same holds for v' ($v' = u'/ [1 + u'^2]^{1/2}$), one shows that (M2) transforms into the following composition law associated with proper velocity

$$u' = u[1 + U^2]^{1/2} + U[1 + u^2]^{1/2} \qquad U = V/ [1 - V^2]^{1/2} \tag{M3}$$

In order to make a link with the method developed all along this work, one sets

$$v' = v + D_v(v,V)V \quad \text{and} \quad u' = u + D_u(u,U)U \tag{M4}$$

and deduces the deviators $D_v(v,V)$ and $D_u(u,U)$ from the substitution of (M4) into (M2) and (M3) leading to

$$D_v(v,V) =\{[v + V]/[1 + vV] - v\}/V \tag{M5}$$

and

$$D_u(u,U) = \{u[1 + U^2]^{1/2} + U[1 + u^2]^{1/2} - u\}/U \tag{M6}$$

Since infinitesimal translations are used, only the expressions associated with V and U tending to zero are needed in the expressions of the extended derivative. Thus, a simple calculation leads to

$$D_v(v, 0) = D_v(v) = 1 - v^2 \quad \text{and} \quad D_u(u, 0) = D_u(u) = [1 + u^2]^{1/2} \tag{M7}$$

Finally the application of the subjective version of the principle of dynamical relativity consists in considering the following second-order differential equations

$$d_z^2E/dz^2 = E, \quad d_z /dz = D_z(z)d/dz \quad z = \{v, u\} \tag{M8}$$

explicitly given by

$$d_v^2E/dv^2 = ([1 -v^2] d/dv\{[1 - v^2] dE/dv\}) = E \tag{M9}$$

$$d_u^2E/du^2 = ([1 + u^2]^{1/2} d/du\{[1 + u^2]^{1/2} dE/du\}) = E \tag{M10}$$

The resolution of these equations accounting for the usual limit conditions, leads to the following expression of energy

$$E = m / [1 - v^2]^{1/2} \quad , \quad E = m [1 + u^2]^{1/2} \tag{M11}$$

As to the second conserved quantity p, the latter is derived as follows:

$$p = d_vE/dv = D_v(v)dE/dv = mv / [1 - v^2]^{1/2} , \quad p = d_uE/du = D_u(u)dE/du = mu \tag{M12}$$

One may verify that the second extended derivative of energy E with respect to v and/or u does not lead to any new conservation law, as required by the dynamical relativity principle. Let us note that v and u are points of view, in the sense that their elimination leads to a unique fundamental relation: $E^2 - p^2 = m^2$ (the essence of Einstein's dynamics). In the forthcoming second part of this Appendix we shall recall the procedure used to derive the above couples of equations associated with the two points of view: v and u.

Link to the Lagrange-Hamilton formalism and to Taylor-Wheeler approach.

The unity provided by this method that leads to both points of view v and u is absent from conventional physics and splits into two different approaches: (i) the Lagrange-Hamilton formulation and (ii) the Taylor-Wheeler approach.

(i) Lagrange-Hamilton Formalism.

One starts by recalling that the Lagrange-Hamilton formalism takes is rooted in the "principle of least action". In the framework of Einstein's relativity the lagrangian is given by

$$L = - m\,[1 - v^2]^{1/2} \qquad \text{(M13)}$$

from which one deduces the two following conservation laws :

$$p = dL/dv = mv / [1 - v^2]^{1/2}, \qquad E = vdL/dv - L = m / [1 - v^2]^{1/2} \qquad \text{(M14)}$$

that corresponds to $(M11)_1$ and $(M12)_1$.

On comparing $p = dL/dv$ with $p = d_vE/dv = D_v(v)dE/dv$, one immediately notices that the first expression of impulse p is defined through a non-physical entity (the Lagrangian) and a non-physical operation (the derivative) while the second results from a physical entity (energy) and a physical operation (the v-derivative or translation operator). Since no recourse is needed to any Lagrangian in the Leibnizian methodology, one is led to a certain "economy of thought". However, this "economy of thought" is paid in return by a "structural complexity" since the translation operator d_v/dv is more complicated to deal with than the simple derivative d/dv. With the use of the velocity concept, the **"economy of thought" is balanced by "structural complexity**".

(ii) Taylor-Wheeler Approach.

Contrary to the Lagrangian which operates on the velocity concept and from which one defines impulse p and energy E through the two expressions given in (M14), the Taylor-Wheeler approach focuses the attention on the invariant kinematical and dynamical elements $\Delta\tau$ and m whose ratio defines both impulse with respect to a space interval and energy with respect to time interval as follows:

$$m/\Delta\tau = p/\Delta r \qquad\qquad m/\Delta\tau = E/\Delta t \qquad \text{(M15)}$$

from which one deduces

$$p = m\,\Delta r/\Delta\tau \qquad\qquad E = m\,\Delta t/\Delta\tau \qquad\qquad (M16)$$

It is worth noting that this approach has a certain Aristotelian flavour in the sense that it uses the Aristotelian reasoning through analogy where (M15) clearly shows that energy is to time ($E/\Delta t$) what impulse is to space ($p/\Delta r$). In addition, none of the four different elements is invariant with respect to inertial systems while the ratios $p/\Delta r$ and $E/\Delta t$ are both invariant and equal to each other. Another specific feature is that this approach avoids infinitesimals when dealing with the basic argument associated with the invariance of $m/\Delta\tau$ since $(m/d\tau)\rightarrow\infty$. However, in the second step that corresponds to the definition of impulse and energy through (M16), it becomes possible to deal with infinitesimals leading to

$$p = m\,dr/d\tau \qquad\qquad E = m\,dt/d\tau \qquad\qquad (M16)$$

(see Ref.[12] page 159). Obviously, on recalling the following definition $u = dr/d\tau$ and Lorentz invariance $dt^2 - dr^2 = d\tau^2$, (M16) takes on the following form

$$p = mu \qquad\qquad E = m[1 + u^2]^{1/2} \qquad\qquad (M17)$$

At this point, one may easily notice the difference between the expressions defined through the Lagrange-Hamilton formalism in (M14) and the definitions obtained from favouring the Newtonian relation $F = ma = m\,du/d\tau = m\,d^2r/d\tau^2$ or its proper time integral $p = mu = m\,dr/d\tau$, (as emphasized by Taylor and Wheeler in Chapter 2 of Ref.[12]).

Appendix N

Key points given in my report entitled "on the foundations of electrodynamics"
(N. Daher, Laboratoire de physique et métrologie des oscillateurs associé à l'université de Franche Comté. Rapport scientifique 1987-1989, pages 7-8).

Summary and translation of the main ideas.
The main ideas are translated and separated into four key points [(i) to (iv)] as follows:
(i) The present work has been developed following interrogations concerning the links between the mathematical elements of a theory and the reality of the physical world….. relative to the laws of **mechanics and electromagnetism**.
After evoking the various works associated with the problem of "infinite energy" in both classical and quantum mechanics through the so-called "renormalisation procedure" the remaining key points are translated as follows:
(ii) This problem (infinite energy) has been approached through classical methods founded on the structure of space-time associated with the field concept. The present approach treats the problem in a quite different manner where *the principles of relativity and causality, without being discarded, do not appear as the cornerstone on which the theory is based*. In a word, **one minimises the role played by space and time giving the primacy to a "finiteness" principle**

without which the very notion of a physical measurable and observable quantity looses its significance.
(iii) This method presents various advantages among which its **mathematical simplicity** and the possibility of a **new interpretation of certain entities** such as mass and charge which are not introduced anymore as basic concepts or fundamental properties. These appear as qualities associated with motion and energy (p: impulse – quantité de mouvement – and E: energy) through **integration constants**.
(iv)**The finiteness principle places in evidence** a new parameter that may be associated with **a microscopic distance**, negligible in the usual experiments of classical electrodynamics. This would explain the validity of Maxwell equations except for very small distances…. In the absence of this parameter, this formulation reduces exactly to the classical relativistic electro-dynamical theory.

Comments and developments of the above presented work.
In order to better understand some of the basic summarized elements given in the above translated report, the attention will be focused firstly on the relation between mechanics and electromagnetism and secondly on the method of differentiation and integration which constitutes the core of the approach.
The passage from mechanical concepts such as energy and impulse (particle like notions) to electromagnetic ones (wavelike notions) may be misleading at first sight, but things become clear when one recalls the correspondences between energy and frequency on the one hand, and impulse and wave number on the other hand. The passage from relativistic dynamics: $E^2 - c^2p^2$ to electromagnetism: $\partial^2/\partial t^2 - c^2\partial^2/\partial x^2$ through $\omega^2 - c^2k^2$ is a well-known fact where one interprets the fundamental equation of relativistic dynamics $E^2 - c^2p^2 = (mc^2)^2 = E_0^2$ as a dispersion relation as follows: $\omega^2 - c^2k^2 = \omega_0^2$. This dispersion relation is associated with the following wave equation:$[\partial^2/\partial t^2 - c^2\partial^2/\partial x^2]A(x,t) = - \omega_0^2A(x,t)$, intimately related to the electromagnetic theory. (These features are developed extensively in Ref.[26], pages 187-193). When these correspondences are kept in mind, the extension of dynamics leads automatically to the extension of electromagnetism as shown in the main text.
In addition to this point, and in order to better grasp the fact that the mass may be associated with an integration constant, one should recall the following basic dynamical third order differential equation: $pp''' + 3\,p'p'' = 0$, whose integration leads to $p^3\,p'' + m^2 = 0$ where m corresponds here to an integration constant. This third order differential equation – understood in this work through the “trans-subjective version of the dynamical relativity principle” (in the particular isotropic case) as shown in the first part of this work – was obtained in a completely different manner. This explains the reason for which the sentence: *“the principles of relativity and causality, without being discarded, do not appear as the cornerstone on which the theory is based”* [given in the key point (ii)] shows that at this epoch the adopted method was not yet related to the dynamical relativity principle. It was associated with a remarkable mathematical property corresponding to Newtonian and Einsteinian dynamics: both of them being expressed through a unique differential form, so that the difference lies in the way one deals with the limit conditions. To show this property, one starts by recalling that Newtonian and Einsteinian dynamics are given by:

$$E = p^2/2m + E_0\,, \qquad E = mc^2(1 + p^2/m^2c^2)^{1/2} \qquad \text{(N1)}$$

whose differentiation leads to

$$dE = [p/m]\, dp \quad \text{(Newton)} \quad dE = [p/m\, (1 + p^2/m^2c^2)^{1/2}]dp \quad \text{(Einstein)} \tag{N2}$$

or equivalently

$$p' = [m/p] = f(p) \quad \text{(Newton)} \quad p' = [m(1 + p^2/m^2c^2)^{1/2} /p] = g(p) \text{ (Einstein)} \;,\; p' = dp/dE \tag{N3}$$

A second derivation leads to

$$p'' = d^2p/dE^2 = [df/dp]\, f(p) = [-m/p^2]\, [m/p] = - m^2/p^3 \tag{N4}$$

for the Newtonian case. The same final result is obtained for the following Einsteinian case:

$$p'' = d^2p/dE^2 = [dg/dp]\, g(p) = [-m/p^2(1 + p^2/m^2c^2)^{1/2}]\, [m(1 + p^2/m^2c^2)^{1/2}/p] = - m^2/p^3 \tag{N5}$$

where the dependence on c, specific of Einstein's dynamics, vanishes by compensation.
The difference between Newton's dynamics and Einstein's one disappears when these dynamics are cast into the above specific differential form. Thus, one is left with the following unique differential equation:

$$p^3 p'' = - m^2 \tag{N6}$$

valid for both Newton and Einstein dynamics. The elimination of the constant associated with the mass concept is obtained by deriving the last equation with respect to E, leading so to

$$pp''' + 3p'p'' = 0 \tag{N7}$$

It is worth noting that the above equation obtained by the differentiation of Newtonian dynamics and Einsteinian one, showing the existence of a unique differential equation that governs both dynamics, is exactly the same as the one derived from the "trans-subjective version of the dynamical relativity principle in the isotropic case, developed in the first part of this work. This fact is essential, since it shows how one may deal with the same structure either in a rather blind mathematical manner or in a clear physical one. This is a typical example where the ***efficiency*** *of* ***mathematics*** *is placed in evidence in dealing with some* ***empirical solutions*** *without being conscious of the* ***physical principle*** *lying behind the mathematical structure.*

The consideration of this last differential equation as a fundamental equation of dynamics leads not only to Newton and Einstein dynamics but, also to other possible dynamical frameworks. The logic behind this procedure runs as follows. Newton's dynamics is characterized, basically, by one constant: the mass (m), while Einstein's dynamics is characterized by two constants: the mass and the coupling constant c, historically associated with the velocity of light (m, c). The third order differential equation contains three constants of integration. These may be associated with the mass, the "velocity of light" and with another coupling constant that one may interpret as a

maximal energy E_M in such a way that energy and impulse remain finite. On letting this maximum energy tend to infinity one recovers Einstein's dynamics. If one adds to this another condition letting the "velocity of light" go to infinity, then one gets Newtonian dynamics. In this new extended picture, Einstein's dynamics appears to be a particular case while Newton's dynamics turns out to be doubly particular.
Recalling that for a maximum energy or impulse corresponds a minimum of duration or length (period or wavelength), one immediately understands why the maximum energy when dealing with dynamics transforms into a minimum length or distance when dealing with wavelike physics. These considerations are given explicitly in the second part of the work.

This Appendix put emphasis at the efficiency of mathematics in dealing with physics. Although I was not conscious of the real significance of the physical principle at work, starting from the unique differential equation (N7), deduced from either Newton's or Einstein's dynamics, I was able to obtain (through an integration procedure) a generalized framework by simply focussing the attention on a possible existence of an upper energy as shown above. Although the different solutions were obtained long ago, only recently these were intimately related to the principle of dynamical relativity and particularly to the its trans-subjective version.

Appendix O

The conventional approach of the history of science.
The history of rational science deals more with what was analytically done by scientists than with what they proposed to do. If the effective analytical work may be compatible with the proposed one, this is not always true. The situation is highly problematical with Leibniz's presentation since it is not always in agreement with what he believes to be basic. **The idea of the best of all possible worlds with an infinite multiplicity of points of view on it, constituting the skeleton of Leibniz's architectonics, is practically absent from the main discussions on dynamics.** [Worse, when it is present, it turns out to be ill-articulated since it is associated with the principle of least action which is logically inadmissible as shown elsewhere in this work]. It is a well-known fact that Leibniz defended the dynamics proposed by Huygens against that of Descartes. But this does not mean that Huygens dynamics seemed to him totally satisfactory, neither that Descartes dynamics was for him intrinsically false, and had to be completely rejected as done by Newton and his followers. If Leibniz prefers the dynamics proposed by Huygens to that of Descartes, it is because **Huygens brought three main advances: (i) he was able to get a well-posed mathematical and physical problem: two equations (two conservation laws) associated with two unknowns (the velocities after a collision). (ii) His dynamics is rationally articulated to the relativity principle. (iii) His approach is compatible with the available experimental results of the 17th century.** In spite of these advances, Huygens parabolic dynamics could be valid only locally, for any regular even function leads at first order to a parabolic form. Leibniz was aware of this. In particular, and at different occasions, he repeated that a parabola may be regarded as an ellipse, one of its foci being cast to infinity. It should also be emphasized that an ellipse is a closed finite figure unlike the parabola which is open to infinity. This finiteness property is compatible with the Leibnizian requirement as to the necessity of a finite velocity. It is remarkable to note, that if one replaces the parabolic Lagrangian (specific of Newton's dynamics compatible with that of Huygens), by an elliptic one, one deduces

automatically a dynamical framework revealing that Huygens dynamics appears as a local one. In addition, the deduced new dynamics subject to the elliptic Lagrangian turns out to be structurally equivalent to the one discovered by Poincaré and Einstein. This clearly shows that in 18th century scientist could easily derive the structure of the 20th century dynamics by simply applying Leibniz's assertion through the use of the Lagrangian formalism: once the Lagrangian is specified, the dynamics becomes automatically determined.

Beyond the analytical framework of conventional physics (and of history of science).

The main Leibnizian features (that the philosophy and history of science could not cope with properly) are those associated with the idea of multiplicity of points of view on a subject matter and that of looking at a reality at different scales. When dealing with the general philosophy of Leibniz, everyone agrees on the importance of these ideas linked to Leibniz's observation across the microscope, as well as his discovery of the different ways one may look at a mechanical curve such as the catenary's one. In addition, every philosopher and historian of science agrees that what Leibniz proposes in his philosophy is intimately linked to his discoveries in mathematics and dynamics and reciprocally. **After telling us that there is a deep unity of thought between the different philosophical, mathematical and dynamical investigations, this unity is broken by the majority of historians when dealing with dynamics. In this discipline intimately linked to Leibniz differential calculus, the ideas of points of view and multiplicity of scales disappear totally from dynamics**. Most scholars operate a huge reduction of Leibniz's methodology, keeping only some analytical features, (for instance the critics addressed against Descartes dynamics in favour of that of Huygens). A few ones tried to reconciliate his philosophy of possible worlds and points of view with his investigations on dynamics, such as L. Bouquiaux [4], but they faced a number of ambiguities and apparently insoluble questions. These questions constituted a basis on which I grounded my investigations. These difficulties in interpretation and the lack of conciliation are due to a dramatic reduction, and to the absence of an appropriate framework (introduction of a new inclusive formal structure going beyond the usual analytical exposition of this approach). **The analytical exposition founded on an exclusive logical framework, unable to exhibit the subtleties of the Leibnizian methodology, led to the belief that Leibniz's approach was contradictory**. This lack of consistency in the examination of Leibniz methodology is due to the difficulty of the problem. First, it requires the introduction of an inclusive logical framework. Then, (with the development of "hyper-specialization"), the philosophers and historians do not possess the adequate mathematical tools necessary to understand the consequences of the critics addressed to Descartes dynamics. In particular, Leibniz was right in noting that if Descartes had realized that his dynamics is false at the origin, he would have accepted Huygens dynamics. The continuous passage from the state of rest to that of motion implies Huygens parabolic form. Huygens dynamics better fits the experiments. The continuity requirement associated with the absolute character of the active substance (satisfied by both Descartes and Huygens dynamics) is sufficient to deduce the parabolic form in the vicinity of the origin. The reason for which Leibniz adheres firmly to the way Huygens establishes the passage from the state of rest to that of motion is not due to empirical reasons, but to rational and necessary ones (provided the principle of continuity has been previously admitted). It is this character of necessity that renders Leibniz so sure of his support to Huygens dynamics. This general result shows that any regularization of Descartes dynamics leads automatically to a mathematical form which is necessarily compatible with that of Huygens in the vicinity of the origin. As for any scale associated with a location not confined to the vicinity of the origin the situation is non-decidable. The regularization of Descartes dynamics proposed by Leibniz without being effectively produced on a mathematical ground, is not unique since there exists an infinite multiplicity of regularized forms as shown in the main text through Eq.(55). According to Leibniz, a dynamics has to be discarded only if one proves its intrinsic falsity on a rational ground: the empirical ground is not safe because a dynamics may be invalid at one scale but valid at another one. As shown in the third part of this work, all who discard Descartes dynamics without discarding the Newtonian one make a logical error since each one is valid at some scale and none is valid at all scales.

Empirical passage from one scale to the other.

The difficulty in dealing with Leibniz methodology is that the methods he proposed were too demanding for his contemporaries. The historical development of physics showed that empirical science passes from one scale to another progressively, with the emergence of a rationality associated with each step. As shown above, one may easily imagine that with Lagrange-Hamilton formalism, it became possible to obtain the structure of 20th century dynamics by the simple replacement of the parabolic Lagrangian by an elliptical one but this did not happen historically. This fruitful formalism, largely applied to 20th century dynamics, did not contribute actively to such a discovery (nor was even proposed). **The sort of rationality that comes after an empirical discovery is by essence partial because such discoveries are attached to a specific or particular mode of measurement** (space over time in the Lagrange-Hamilton formalism). On favouring one point of view, one misses the main features of the Leibnizian idea of integral rationality including a multiplicity of points of view on a given reality (here dynamics). **Such an inclusive logic lies at the basis of the conciliatory attitude of Leibniz with respect to the different dynamics present at his epoch** developed independently and with different methods by Descartes, Huygens and Newton. It is worth recalling that the simple conciliation of Descartes dynamics with that of Huygens could have led to 20th century dynamics through the principle of relativity; and even without it, if one starts with the Eq.(55) (including an infinite number of potentialities). The application of the principle of simplicity as shown in the main text leads to the same result. Such **multi-rationality** requires necessarily an **inclusive logical framework,** the absence of which leads automatically to misunderstandings and ambiguities. The lack of such a framework led to believe that Leibniz's assertions on dynamics are not only metaphysical but also contradictory, as already noted. One of the main goal of the present work is to show that these assertions are neither metaphysical nor contradictory; they simply require a formal **language of a higher level** capable of dealing simultaneously with such a multiplicity. Leibniz's conciliatory attitude was wrongly attributed to his lack of knowledge and his incapacity to distinguish the true from the false. The present work clearly shows that Leibniz was right in foreseeing that a profound thought on motion and substance is not **an easy** question, (easy enough to be answered by one or another scholar of his epoch). It is a question whose solution may require the intelligence of numerous minds over centuries. This explains why Leibniz privileged the development of methods apt to exhibit what is unseen directly, as the account for unlimited degrees of freedom so that what is impossible to deal with at his epoch became possible at a later time, when new technical developments allowed to see what was previously impossible to experiment. One main reason behind his belief is due to his observation across the Leeuwenhoek microscope that showed him things he could not imagine. **Leibniz's claim asserting that he**

prefers Leeuwenhoek who tells him what he sees on a Cartesian who tells him what he thinks [5] is significant as to Leibniz epistemology and philosophy of nature.
According to Leibniz, and contrary to what was believed at his epoch, science was still in its infancy and much remains to be done before acquiring a good understanding of dynamics. There is a great difference between Lagrange's belief that the laws of the universe have been found by Newton and the only remaining thing to do is to put all this in a rational framework, and Leibniz's belief that science never ends, and that actual knowledge is a local imprint of a global one that remains to be developed across the coming centuries. This attitude explains why Leibniz proposes such a general framework, apt to encompass an unlimited number of points of view on a given reality (here dynamics through positive active substance $S(x) > 0$). One of the main problems of men, according to Leibniz, is that they **confuse** their own **point of view** on a thing with the **thing** itself; the imposition of one point of view, the one adopted by Newton and followed by the majority of physicists, (embedded in a coherent mathematical framework) leads only to a partial rationality and not to an integral multiple one.

<u>*Leibniz's conception of dynamics.*</u>
Let us see now in more detail how such ideas apply positively to dynamics. To this end, let us start by considering the question of the "vis viva" or "active force" (kinetic energy in modern terms). After having called the Huygens entity associated with the positive active substance $[g(v) = mv^2]$ the active force, **Leibniz introduces the notion of "absolute force" that corresponds mathematically to $f(x) = f(-x)$.** If one adopts one of the cornerstones of Leibniz methodology relative to the multiplicity of points of view, then one is led to a double extension: The first of these consists in replacing the definition of motion by some yet undefined parameter x (not necessarily associated with the velocity concept: ratio of a length over a duration); the second consists in replacing the parabolic character by some general undetermined form. Only when these two kinds of weak forms are considered, one is able to include in this more general framework both Huygens system ($h(v)=mv^2$), and Descartes one ($g(u)=m|u|$) associated with the positive active substance (called by Descartes "quantity of motion"). The difference lies not only in the consideration of two different functions h and g, but also in the consideration of two different definitions of motion: v for Huygens and u for Descartes. Motion is defined in relation to "active substance" by the above mentioned symmetry requirement $f(x) = f(-x)$ verified simultaneously by h and g where substance is always positive and invariant with respect to the inversion of motion. This property of invariance under inversion of sign (reflection), satisfied by u and v simultaneously, implies a relation between u and v where the inversion of v leads to the inversion of u. Thus, the basic properties of the different points of view on motion consists in recognizing the necessary existence of an order between the different points of view, so that the inversion of sign of any point of view leads automatically to the inversion of all other points of view. When the idea of points of view is not considered (as in conventional physics) the different distinctions vanish and no conciliation is possible anymore. If one identifies u with v associating it with the velocity concept as defined through the Lagrange-Hamilton formalism, then one leads automatically to the conclusion usually admitted. Huygens dynamics is conceptually "accidental" although mathematically equivalent to the Newtonian one, while Descartes dynamics is intrinsically false. One may refer to the third part of this work, where all this is examined at length.

After this consideration, let us note that Leibniz's emphasis, through his notion of "absolute force", on the symmetry property: $f(x) = f(-x)$ associated with the "active substance", and its correlated form associated with motion leading to $m(-x) = - m(x)$, [m for motion, not to confuse with the mass] constitutes a qualitative way that deals with motion only partially since such a property does not allow to define motion quantitatively. (Notice however, that this global constraint is already sufficiently strong to account for the rest state in a given reference frame. Indeed, if one considers an analytic continuous framework through regular functions well defined at the origin, then one shows that the state of rest is one, while the state of motion is infinitely multiple. This is due to the following property: $m(-x) = - m(x)$ implying automatically $m(0) = 0$ whatever the function m associated with the different points of view on motion and whatever the point of view x provided the symmetry argument (odd character) is satisfied. In the main text the different points of view as well as the different functions that operate on each one of them are distinguished by use of Greek indices. This is not essential here since as long as one does not look for a specific order correlating the different points of view with each other the introduction of indices complicate the expressions uselessly).
Let us conclude this Appendix by noting that we do not claim that Leibniz was explicit concerning all these points but if one orders the different qualitative assertions given by Leibniz on dynamics and in agreement with his basic idea as to the existence of different perspectives on a given reality (here motion), then, one is led automatically to these considerations. Obviously, one needs to perform a deeper research on Leibniz's methodology to see to what extent, Leibniz was conscious of all these facts. In this regard, let us recall that according to historians and philosophers specialized in Leibniz's approach of nature, they still need about half a century to study the writings produced by Leibniz, still unexamined. It would be possible to discover in the near future some formal and quantitative confirmation of the qualitative Leibnizian ideas that we formalize and quantify in the present work.

Appendix P

Physical justification of an analogy between the oscillator problem and fundamental physics.
This Appendix is devoted to an early intuition concerning the possibly fruitful analogy between the oscillator for conservative and dissipative systems and fundamental physics. In the course of mechanics delivered in the seventies by one of my professors P. Brousse, [32] Chapter 16 is entitled "oscillations". Different analogies are performed between mechanical and electrical systems dealing with displacement parameter (spring), rotation angle (torsion) and electronic charge (electric circuit). Noting that the expressions are given through trigonometric functions (cos, sin) deduced from the oscillator second order equation ($y'' + b^2y = 0$) and that the passage to hyperbolic functions (cosh, sinh) is a simple question of sign ($y'' - c^2y = 0$), it appeared to me interesting to examine if the analogy could be extended somehow to fundamental physics. The hyperbolic solution is a characteristic feature of Einstein's relativity theory. Beyond this simple analogy lies the idea of a possible generalization of relativity theory, following the line of thought developed by the generalization of the above mentioned differential equation to "damped" oscillations ($y'' + ay' + b^2y = 0$), leading thus to a formally similar equation ($y'' + ay' - c^2y = 0$) except for the already mentioned sign. As it will be shown in the forthcoming development, such an extension leads to a remarkable conclusion concerning the metrical structure of presently

available physical theories. More precisely, **the Euclidean, hyperbolic and Riemannian metrics at the basis of Newtonian physics and Einsteinian special and general relativities turn out to be too narrow to encompass the obtained solutions**. In spite of its conceptual interest (it leads to new inquiries about the laws of nature), this mathematically remarkable result does not appear as physically justified on a rational ground : the "damped" hyperbolic solution that may be associated with dynamics lacks physical justification. Obviously, the present work provides such a justification, since the above equation is **compatible with the dynamical relativity principle** (as shown below).
In order to see the influence of "damping" on the metrical structure and its compatibility with the dynamical relativity principle, let us start with the dynamical equations. An associated kinematics can be deduced through the introduction of a specific space-time structure following a well-known method due to Taylor and Wheeler. To this end, we set

$$E = E_0 \exp(Sw) \cosh(w) \qquad \text{(P1)}$$

$$p = E_0 \exp(Sw) \sinh(w) \qquad \text{(P2)}$$

where E and p satisfy the following differential equations:

$$d^2E/dw^2 - 2SdE/dw - (1 - S^2)E = 0 \qquad \text{(P3)}$$

$$p = dE/dw - SE \qquad \text{(P4)}$$

Notice the similarity with Eqs.(35) directly deduced from the application of the dynamical relativity principle. The distinction between (E,p) given above and (e,P) given in Eqs.(35), is of no importance in so far as the dynamical relativity principle and conservation laws are concerned. Here S corresponds to the damped contribution associated with the hyperbolic structure, derived by analogy with the damped oscillator equation. In dynamics, this corresponds to an anisotropic effect (broken parity) whose absence, (S = 0) leads to Einstein's dynamics expressed with the rapidity parameter the elimination of which leads to the fundamental hyperbolic equation of Einstein's dynamics: $E^2 - p^2 = E_0^2$. (using natural units : c = 1):

$$E = E_0 \cosh(w) \qquad \text{(P5)}$$

$$p = E_0 \sinh(w) \qquad \text{(P6)}$$

In order to see that these equations are compatible with the principle of dynamical relativity one may refer to the main text and particularly to the second part of this work associated with anisotropy or broken parity. In order to establish a direct link with the velocity concept at the basis of space-time physics, we consider the ratio between impulse and energy as follows:

$$v = p/E = dx/dt \qquad \text{(P7)}$$

from which one deduces

$$v = \tanh(w) \qquad \text{(P8)}$$

It is remarkable to note that in spite of the difference between the present solution and Einstein's dynamics **one gets the same law for the composition of motion** $v' = (v + V)/(1 + vV)$.
The "damping" effect is eliminated by compensation. This is a direct consequence of $\tanh(w+W) = (\tanh w+\tanh W)/(1+\tanh w \tanh W)$.

Let us recall that the velocity is usually, defined by $v = p/E = dE/dp$ in the framework of Einstein's dynamics. Here, one gets two different velocities $v = p/E$ as shown above and $\hat{u} = dE/dp$. Only the first of these (v) satisfies the usual law of composition of motion. The other one ($\hat{u}$) leads to a more complicated expression, which reduces to (v) only when $S = 0$. (Let us emphasize the fact that we are not looking here for a complete approach, we are simply drawing the attention on some remarkable properties which deserve to be revealed especially that they lead to relatively simple solutions that break Lorentz metric extending it to scale laws as shown below). The Lorentz metric at the basis of most presently available theories will be recovered when the scale parameter S vanishes. Let us start by expressing energy and impulse in terms of the velocity v instead of the rapidity w, then one gets:

$$E = [(1+v)/(1-v)]^{S/2}\ E_0/[1-v^2]^{1/2} \qquad \text{(P9)}$$

$$p = [(1+v)/(1-v)]^{S/2}\ E_0 v/[1-v^2]^{1/2} \qquad \text{(P10)}$$

The elimination of v leads to the following dynamical relation

$$(E - p)^{1+S} (E+ p)^{1-S} = E_0^{\,2} \qquad \text{(P11)}$$

where it is immediately shown that for S=0 one recovers the fundamental relation of Einstein's dynamics. The passage to the space-time structure may be obtained following the line of thought developed by Taylor and Wheeler (given in Appendix J),where one sets

$$p = E_0 dx/d\tau \qquad E = E_0 dt/d\tau \qquad \text{(P12)}$$

The substitution of (P12) into (P11) leads to the following metrical structure

$$(dt - dx)^{1+S} (dt + dx)^{1-S} = d\tau^2 \qquad \text{(P13)}$$

or equivalently to

$$[dt^2 - dx^2] [(dt - dx)^2/(dt^2 - dx^2)]^S = d\tau^2 \qquad \text{(P14)}$$

This last expression may be extended to a three dimensional framework as follows:

$$[dt^2 - \mathbf{dx}^2] [(dt - \mathbf{n.dx})^2/(dt^2 - \mathbf{dx}^2)]^S = d\tau^2 \qquad \text{(P15)}$$

where one discovers a direct connection with another work [33]. In a four dimensional framework this corresponds to the following form

$$[\eta_{\mu\beta}\, dx^{\mu}\, dx^{\beta}]\, [(\eta_{\mu\beta}\, n^{\mu} dx^{\beta})^2/\eta_{\mu\beta}\, dx^{\mu}\, dx^{\beta}]^{S} = d\tau^2 \qquad \text{(P16)}$$

that goes beyond the usual geometries used in physics (Euclidean, hyperbolic and Riemannian) leading to so-called Finsler geometry.

The above relation reduces to Einstein's space-time metrical framework (corresponding to Lorentz metric: $dt^2 - dx^2 = d\tau^2$) when the exponent S is set equal to zero or neglected with respect to unity. Moreover, one can notice that this kind of metrical structure does not enter into the mould of any of the three well-known physical metrics: Euclidean, hyperbolic and Riemannian ones. These metrics are expressed usually through the following quadratic form $ds^2 = g_{\mu\beta}\, dx^{\mu}\, dx^{\beta}$ where the choice of the coefficients $g_{\mu\beta}$ determines one of the three above mentioned geometries. Even if one ignores whether these results, initially obtained through the analogy with the study of the "oscillator", will be experimentally verified, one has more confidence in their plausibility because of their compatibility with the dynamical relativity principle in its weak form (where the isotropy requirement is not imposed). When I firstly made these calculations I was not aware of their link to the relativity principle. This awareness provides them certain legitimacy. It is also worth noting that **these analogies and relations compatible with the principle of relativity are closer to physics than many adopted mathematically oriented extensions and generalizations.** Indeed, as well-known from geometric and algebraic considerations, the mathematical framework provides numerous possible extensions, some of them being physically inadmissible for they do not respect the relativity requirement. More precisely, if the different geometries used in space-time physics associated with the standard model turn out to be insufficient as recognized by many modern physicists, then these should be extended. Such an extension – already mentioned by Riemann himself as being possibly necessary for physics – leads to so-called **Finsler geometry**. This framework is wider than Riemann's one since it includes the metrical structure as a particular case, but there is no direct connexion between such a wide framework including numerous possibilities and the principle of dynamical relativity. This is the main reason for starting with a dynamical framework rather than with a geometrical one. The latter leads either to very narrow physical frameworks as shown by conventional physics (above-mentioned geometries) or to very wide ones, from which it is not easy to select the one to be associated with physics (leading to the impossibility of producing a predictive physical framework). Usually, physicists who pursue the approach initiated by Newton – extended by Einstein where physics is founded on geometry – face the problem of a certain chaos resulting from the multiplicity of possible forms. This chaos is mastered or controlled by the imposition of a principle of simplicity. An example of the use of such a principle has been given in the main text where it is shown how the latter operates leading to remarkable results. However, **if such geometrical results obtained through a simplicity criterion, may lead to efficient solutions associated with physics, these remain mysterious as long as one does not grasp the underlying physical principles. Only then, one may deal with clear-sighted physics where explanation and exploration complement each other in a constructive manner.**

Appendix Q

Anisotropy or broken parity: apparent and real.

This appendix is devoted to the distinction between "apparent" and "real" anisotropy or broken parity. The first of these appear when dealing with integration constants associated with a differential equation. The second is inherent to the differential equation itself. Usually, the term "isotropy" takes its origin in the notion of space. Since the present approach is mainly dynamical, it would be more appropriate to talk of "broken parity" than "anisotropy". This is not only motivated by the absence of the notion of space from the beginning but also by the possibility of another possible interpretation associated with "parity" such as differential aging reversal dealt with in Ref.[34]. Logically speaking, one may say that if isotropy implies parity the reciprocal is not true.

When dealing with Einstein's dynamics, following the line of thought developed by the "emergent rationality" through the rapidity parameter, one is led to the following fundamental second order differential equation.

$$d^2E/dw^2 = E \tag{Q1}$$

where we have used the natural units c = 1. The solution of this equation may be written as follows:

$$E = a \cosh w + b \sinh w \tag{Q2}$$

Since impulse derives from energy with respect to rapidity,

$$p = dE/dw \tag{Q3}$$

one gets

$$p = a \sinh w + b \cosh w \tag{Q4}$$

It is easily shown that if one does not impose any restriction on the integration constants, then energy and impulse are neither even nor odd as usual where one has

$$E = a \cosh w \qquad\qquad p = a \sinh w \tag{Q5}$$

Since in 1+1 dimension, isotropy is accounted for through the **even character of energy** and/or the odd character of impulse, on imposing the following global constraint:

$$E(w) = E(-w) \qquad \text{or} \qquad p(w) = -\,p(-w) \tag{Q6}$$

the integration constant b vanishes (b=0) and Eqs. (Q2) et (Q4) reduce to (Q5). This reduction could have been obtained by the following local constraint:

$$w = 0 \ ,\ p = 0 \tag{Q7}$$

It is worthy of note that, here, the **global** constraint (Q5) leads exactly, to the same result as the **local** one (Q7). This fact characterizes the “apparent” anisotropy of (Q2) and (Q4) as it will be explained later on. Before dealing with this, let us explain what is meant by “real” anisotropy. To this end, one proceeds by replacing (Q2) and (Q4) by

$$E = \exp(Sw)\,[a \cosh w + b \sinh w] \tag{Q8}$$

$$p = \exp(Sw)\,[a \sinh w + b \cosh w] \tag{Q9}$$

It is important to realize that in this case, the local and global constraints lead to different results. The local constraint (Q7) is weaker since it leads to

$$E = \exp(Sw)\,[a \cosh w] \tag{Q10}$$

$$p = \exp(Sw)\,[a \sinh w] \tag{Q11}$$

while the global (Q6) one leads to (Q5). In spite of the fact that Eqs.(Q8)-(Q11) do not verify Eq.(Q1) these solutions remain compatible with the dynamical relativity principle, since the inertia (second derivative of energy) is equal to a linear combination of energy and its first derivative or of energy and impulse,

$$d^2E/dw^2 = (1 - S^2)E + 2S\,dE/dw = (1 + S^2)E + 2Sp \tag{Q12}$$

Notice that impulse is here associated with the derivative of energy to which one adds a proportionality relation with respect to energy:

$$p + SE = dE/dw \tag{Q13}$$

This renders the equations simple without any lack of generality, since if the couple (E, dE/dw) corresponds to conservation laws, then the other couple (E, $p=dE/dw+SE$) constitutes an equivalent system. If A, B correspond to two conservation laws then any linear combination is also a conservation law so that one has some degrees of freedom in choosing the couple of laws to be considered. On applying this property to (Q2) and (Q4) then one may eliminate the constant of integration b without the use of the local or global constraints given through (Q6) and (Q7). Indeed, on using the following combinations:

$$E' = [a/(a^2-b^2)][aE - bp] \qquad p' = [a/(a^2-b^2)][ap - bE] \tag{Q14}$$

The couple of conservation laws (E, p) transforms into an equivalent couple (E', p') verifying

$$E' = a \cosh w \qquad p' = a \sinh w \tag{Q15}$$

which is similar to (Q5). Here lies the main reason for which Eqs.(Q2) and (Q4) are associated with apparent anisotropy while Eqs.(Q8) and (Q9) include both the apparent and the real anisotropies through the constants b and S. These are of different natures, since one is associated with an integration constant while the second is inherent to the differential structure. It is present not only in the solution but also in the differential equation.

Notice that, if one shows that certain constants of integration may play a major role in the passage from an infinite energy to a finite one, one should recognize that this role played by the integration constants is not adapted to the distinction between isotropy and anisotropy: it may introduce an apparent anisotropy which complicates the expressions without having a true physical relevance.

Appendix R

Splitting of the velocity concept into two distinguishable concepts when the hyperbolic character is broken. (From dynamics to kinematics and reciprocally).

In this Appendix we shall focus the attention on two sorts of velocities which are not distinguished in Einsteinian and Newtonian dynamics. Indeed, let us recall that one may write, (in natural units c = 1)

$$v = p/m = dx/dt \qquad v = p/M = dx/dt \qquad M = E, \tag{R1}$$

where the Einstein's framework may be split as follows

$$v = (p/m)(m/M) = (dx/d\tau)(d\tau/dt) \tag{R2}$$

showing that in Newton's physics there is no distinction between M and m as well as between dt and $d\tau$ (absolute time). This definition of the velocity in relation to dynamics and kinematics is also compatible with the first Hamilton canonical equation $v = dE/dp$. This fact leads automatically to the parabolic and hyperbolic structures that characterize the Newtonian and the Einsteinian isotropic frameworks. In particular, and in the non degenerate case associated with Einstein's physics, the passage from kinematics to dynamics may be obtained as follows. Having discovered from electromagnetism that the structure of space-time may be written as follows: $dt^2 - dx^2 = d\tau^2$, one may show that the structure of dynamics is obtained by simply noting that time is to energy what space is to impulse so that one is left with the following dynamical structure: $E^2 - p^2 = m^2$. This procedure, justified elsewhere in this work and associated with the method proposed by Taylor and Wheeler [12], may be applied starting from kinematics towards dynamics or from dynamics towards kinematics. We shall use this same procedure to discover the structure of kinematics beginning with dynamics deduced from the present Leibnizian dynamical relativity principle. Then we shall go back from kinematics to dynamics by adopting the "usual rationality" through the Lagrange-Hamilton formalism. In proceeding in this manner, we deal with two different methods that apply in a conventional isotropic framework, leading to the same results while the latter would be different in reason of anisotropy or broken parity. In particular, this will lead us to two different notions of velocity. These coincide in the Newtonian and Einsteinian frameworks since $v = p/M$ with $M = m$ (Newton) and $M = E/c^2 = E$ (Einstein) but do not coincide anymore with the first canonical Hamilton equation dE/dp. To see this, let us start with the following dynamical structure:

$$(E - p)^{1+S} (E + p)^{1-S} = E_0^2 \quad \text{(R3)}$$

derived in Appendix P to which one associates the following kinematical structure:

$$(dt - dx)^{1+S} (dt + dx)^{1-S} = d\tau^2 \quad \text{(R4)}$$

On applying the usual procedure associated with the Lagrange-Hamilton formalism:

$$A = - m \int d\tau = \int L\, dt \quad \text{(R5)}$$

one deduces the Lagrangian

$$L = - m (1 - v)^{(1+S)/2} (1 + v)^{(1-S)/2} \quad \text{(R6)}$$

From which one deduces impulse π and energy ε by use of the conventional procedure

$$\pi = dL/dv \qquad \varepsilon = v dL/dv - L = v\pi - L \quad \text{(R7)}$$

so that one is left with the following relations:

$$\pi = dL/dv = [(1 - v) /(1 + v)]^{S/2}[(v + S)/ (1 - v^2)^{1/2}] \quad \text{(R8)}$$

$$\varepsilon = v dL/dv - L = [(1 - v) /(1 + v)]^{S/2}[(1 + Sv)/ (1 - v^2)^{1/2}] \quad \text{(R9)}$$

On taking the ratio between impulse and energy as follows one obtains:

$$\nu = \pi / \varepsilon = (v + S)/(1 + Sv) = (p + SE)/(E + Sp) = dE/dp \quad \text{(R10)}$$

One easily notices that the two definitions of the velocities $v = p/E$ and $\nu = \pi / \varepsilon$ reduce to a unique concept in the absence of anisotropy since one gets

$$v = p/E = \nu = \pi / \varepsilon = dE/dp \quad \text{when } S = 0 \quad \text{(R11)}$$

One may also deduce the following relations

$$\pi = p + SE \qquad \varepsilon = E + Sp \quad \text{(R12)}$$

which show that if the couple (p, E) corresponds to two conservation laws, then the couple (π, ε) constitute two equivalent laws since any linear combination of p and E leads to equivalent results.
On expressing the dynamical structure in terms of the couple (π, ε) instead of (p, E) one gets

$$(\varepsilon - \pi)^{1+S}\,(\varepsilon + \pi)^{1-S} = \varepsilon_0^{\,2} \qquad \text{(R13)}$$

where we have set

$$\varepsilon_0^{\,2} = E_0^{\,2}\,[(1 + S)/(1 - S)]^S/[1 - S^2] \qquad \text{(R14)}$$

Notice the perfect formal analogy between (R13) and (R3).

Appendix S

Explicit solution associated with the trans-subjective version of the principle of dynamical relativity.

Let us recall that the trans-subjective version of the principle of dynamical relativity may be expressed in the following form:

$$p\,dp/dE = \lambda E + \gamma\, p + \eta \qquad \text{(S1)}$$

or equivalently by

$$p\ dp/dR = \lambda R + \gamma\, p, \qquad R = E + \eta/\lambda \qquad \text{(S2)}$$

On setting

$$z = p/R \qquad p' = dp/dR \qquad \text{(S3)}$$

one may establish the following identity

$$R\ dz/dR = dz/\,dx = p' - z \qquad \text{(S4)}$$

where we have set

$$x = \mathrm{Ln}(|R|) \qquad \text{(S5)}$$

The combination of (S3) with (S2) leads to

$$p' = \lambda/z + \gamma \qquad \text{(S6)}$$

The elimination of p' between the Eqs.(S6) and (S4) yields

$$dz/dx = \lambda/z + \gamma - z \qquad \text{(S7)}$$

This allows one to write the following integral form

$$x = \int dz/\,[\lambda/z + \gamma - z] = \int z dz/\,[-z^2 + \gamma z + \lambda] \qquad \text{(S8)}$$

whose solution corresponds to

$$x = -\{ \tfrac{1}{2} \mathrm{Ln}(|-z^2 + \gamma z + \lambda|) + (\gamma/\mu)\, \mathrm{Arctanh}[(\gamma - 2z)/\mu] + K\} \qquad (S9)$$

with

$$\mu = [4\lambda + \gamma^2]^{1/2} \qquad (S10)$$

On accounting for the expressions of x, R and z given respectively in (S5), (S2) and (S3) one is left with

$$z = p/R = \{p/(E + \eta/\lambda)\} \qquad (S11)$$

and

$$\mathrm{Ln}(|R|) = -\{ \tfrac{1}{2} \mathrm{Ln}(| -(p/R)^2 + \gamma (p/R) + \lambda|) + (\gamma/\mu)\, \mathrm{Arctanh}[[\gamma - 2(p/R)]/\mu] + K\} \qquad (S12)$$

The isotropic physical formulations correspond to $\gamma = 0$. Thus one is left with

$$\mathrm{Ln}(|R|) = -\{ \tfrac{1}{2} \mathrm{Ln}(| -(p/R)^2 + \lambda|) + K\} \qquad (S13)$$

Or equivalently

$$|R| = C|R|/[\lambda R^2 - p^2]^{1/2} \qquad (S14)$$

On account of (S2) one deduces

$$[\lambda R^2 - p^2]^{1/2} = [\lambda(E + \eta/\lambda)^2 - p^2]^{1/2} = C \qquad (S15)$$

By an adequate choice of the constants η, λ and C, one may obtain the solutions associated with "doubly or deformed special relativity". Obviously, when η vanishes one is led to

$$\lambda E^2 - p^2 = C^2 \quad \Leftrightarrow \quad E^2/c^2 - p^2 = m^2c^2 \qquad (S16)$$

where one recognizes the structure of Einstein's dynamics provided one chooses the constants in the following way

$$\lambda c^2 = 1 \qquad \text{and } C^2 = m^2c^2. \qquad (S17)$$

Appendix T

Classification of the different finite and uneven (broken parity) solutions: including recent empirically or mathematically oriented approaches.

The present Appendix is devoted to the physical study of the general solution derived from the trans-subjective version of the principle of dynamical relativity (mathematically developed in Appendix S). Since the present approach is general accounting for two different unusual effects: the finiteness of energy ($E < E_M$) and the broken parity [$F(E, p) \neq F(E, -p)$]), it is important to uncouple the different situations, showing the relevance of each contribution and its relation to recent approaches of dynamics. In order to justify the different coupling constants, let us recall that the second order differential system of equations including an infinite multiplicity of points of view on motion, which constitutes the starting point of the methodology, may be transformed into a first order differential equation. This transformation eliminates the multiplicity of points of view in favour of a fundamental differential equation, dealing only with the conserved entities: energy and impulse. As shown in the main text, this first order equation includes three constant coefficients, whose justification is given by the properties of conservation laws. A fourth one occurs as a constant of integration. These four constants will play a major role in the distinction to be made between the different approaches. In order to get a structure as close as possible to the fundamental equation of Einstein's dynamics: $[E^2/c^2 - p^2]^{1/2} = mc$ let us take into account the following identity

$$\mathrm{Arctanh}(x) = \tfrac{1}{2} \mathrm{Ln}|(1+x)/(1-x)| \qquad (T1)$$

which allows to transform Eq.(S12) as follows:

$$\{[(\mu+\gamma)R - 2p]/[(\mu-\gamma)R + 2p]\}^{\gamma/2\mu} [\lambda R^2 + \gamma Rp - p^2]^{1/2} = C \qquad (T2)$$

with

$$R = E + \eta/\lambda \qquad \mu = [4\lambda + \gamma^2]^{1/2} \Leftrightarrow \lambda = [\mu^2 - \gamma^2]/4 \qquad (T3)$$

Or equivalently,

$$\{[\mu_\gamma^+ R - p]/[\mu_\gamma^- R + p]\}^{\gamma/2\mu} [\mu_\gamma^+ \mu_\gamma^- R^2 + \gamma Rp - p^2]^{1/2} = C \qquad (T4)$$

where we have set

$$\mu_\gamma^+ = (\mu+\gamma)/2, \qquad \mu_\gamma^- = (\mu-\gamma)/2 \qquad (T5)$$

In the absence of anisotropy the coefficients μ_γ^+ and μ_γ^- become equal and reduce to λ as follows:

$$\mu_0^+ = \mu_0^- = \mu/2 = [\lambda]^{1/2} \qquad (T6)$$

Noting the following identity

$$[\mu_\gamma^+\mu_\gamma^- R^2 + \gamma Rp - p^2]^{1/2} = \{[\mu_\gamma^+R - p][\mu_\gamma^- R + p]\}^{1/2} \quad \text{(T7)}$$

the above equation may be transformed so that one is finally left with the following form:

$$\{[\mu_\gamma^+R - p]^{1+\gamma/\mu} [\mu_\gamma^- R + p]^{1-\gamma/\mu} = C^2 \quad \text{(T8)}$$

This general equation (expressed in different forms), accounts for anisotropy through the coefficient γ and for finiteness through the coefficient η [or more precisely through adequate combinations between η and λ, the latter reducing to the inverse of the square of the light velocity when η vanishes as shown above in Appendix S through (S17)]. Among the three coefficients, γ, η and λ that occur in the trans-subjective postulate of dynamical relativity given in (S1), only the last one is well-known. It appears in the passage from Newtonian ($\lambda = 1/c^2 \to 0$) to Einsteinian dynamics where this inexistent coefficient acquires a finite existence. Since the advent of Einstein's dynamics and until today, one may say that the most basic approaches of physics including classical and quantum considerations account neither for γ (anisotropy or broken parity with respect to v) nor for η (finiteness of energy). Since the above equation accounts for both anisotropy and finiteness, it may be useful to distinguish between **two intermediate results** corresponding respectively to the two following couples: (i) anisotropy or broken parity characterized by $(\gamma, \eta) = (\gamma, 0)$ and (ii) finiteness associated with $(\gamma, \eta) =(0, \eta)$.

(i) anisotropy or more generally broken parity : $(\gamma, \eta) = (\gamma, 0)$.
It is immediately checked out that when η vanishes, then R reduces to E as shown in (T3) but the structure of (T2) and (T4) to (T8) remains the same. Thus, one is led to the following intermediate result:

$$\{[(\mu + \gamma)E - 2p]/ [(\mu - \gamma)E + 2p]\}^{\gamma/2\mu} [\lambda E^2 + \gamma Ep - p^2]^{1/2} = C \quad \text{(T9)}$$

that may also be expressed in the two following ways:

$$\{[\mu_\gamma^+E - p]/ [\mu_\gamma^- E + p]\}^{\gamma/2\mu} [\mu_\gamma^+\mu_\gamma^- E^2 + \gamma Ep - p^2]^{1/2} = C \quad \text{(T10)}$$

and

$$\{[\mu_\gamma^+E - p]^{1+\gamma/\mu} [\mu_\gamma^- E + p]^{1-\gamma/\mu} = C^2 \quad \text{(T11)}$$

as shown in (T4) and (T8). Notice that one should set

$$\mu_0^+ = \mu_0^- = \mu/2 = [\lambda]^{1/2} = 1/c \quad \text{and} \quad C = mc$$

if one wishes to obtain Einstein's dynamics as a limit case in the absence of anisotropy or broken parity.
(ii) finiteness of energy where parity is satisfied : $(\gamma, \eta) =(0, \eta)$.
In this second intermediate situation R remains different from E but the basic equation will be much simplified since the three different but equivalent forms reduce to the following one

$$[\lambda R^2 - p^2]^{1/2} = C \quad \Leftrightarrow \quad [\lambda R^2 - p^2] = C^2 \tag{T12}$$

or equivalently, after having expressed R in terms of E by use of (T3):

$$[\lambda(E + \eta/\lambda)^2 - p^2]^{1/2} = C \quad \Leftrightarrow \quad [\lambda E^2 + 2\eta E - p^2] = K\,, \quad K = C^2 - \eta^2/\lambda \tag{T13}$$

Newton and Einstein dynamics correspond to $\lambda = 0$ (parabola) and $\eta = 0$ (hyperbola).

Important remark concerning the finiteness requirement linked to recent empirically oriented and partially rational approaches.

At first sight, one may think that the couple of coefficients (γ, η) characterizes anisotropy and finiteness respectively. However, a closer look at the structure shows that one may deal with finiteness even when the two coefficients vanish: $(\gamma, \eta) = (0, 0)$. In spite of the structural analogy with Einstein's dynamics where one gets

$$[\lambda E^2 - p^2]^{1/2} = C \tag{T14}$$

one should recall that the constants λ and C may be chosen arbitrarily. Thus, only when the first constant λ is associated with $1/c^2$ and the second C with mc, that Einstein's dynamics is recovered. One may choose other values for λ and C. In particular, if one replaces $\lambda = 1/c^2$ by another value such as $\lambda = 1/c^2\{1 + (mc^2/ E_M)^2\}$ which reduces to $1/c^2$ when the maximal energy E_M tends to infinity, then this simple replacement casts Eq.(T6) into

$$[1/c^2\{1+(mc^2/ E_M)^2\}E^2 - p^2]^{1/2} = mc \tag{T15}$$

or equivalently to

$$e^2/c^2 - P^2 = m^2c^2\,, \qquad e = E / [1 - (E/E_M)^2]^{1/2} \qquad P = p /[1 - (E/E_M)^2]^{1/2} \tag{T16}$$

Here, one recognizes the approach of the so-called "canonical doubly special relativity" given in Ref.[8] where energy is precisely bounded. However, if one refers to the other approach given in Ref.[7], then, one should account for the intermediate solution associated with $(\gamma, \eta) = (0, \eta)$ where the coefficient η does not vanish and where the finiteness requirement is obtained by a sort of combination between the two constant coefficients, λ and η. To see this, let us recall that the solution proposed in Ref.[7] corresponds to

$$e^2/c^2 - P^2 = m^2c^2\,, \qquad e = E / [1 - (E/E_M)] \qquad P = p /[1 - (E/E_M)] \tag{T17}$$

or equivalently to

$$E^2/c^2 - p^2 = m^2c^2\,[1 - (E/E_M)]^2 \tag{T18}$$

also written as follows:

$$[1/c^2\{1-(mc^2/E_M)^2\}E^2 + 2(m^2c^2/E_M)E - p^2] = m^2c^2 \qquad (T19)$$

This last expression does not enter into the too particular case $(\gamma, \eta) = (0, 0)$ but it belongs to the intermediate class of solutions associated with the couple $(\gamma, \eta) = (0, \eta)$ directly related to (T13). A simple identification procedure leads to the following values:

$$\lambda = 1/c^2\{1-(mc^2/E_M)^2\}, \quad \eta = m^2c^2/E_M \qquad C^2 = m^2c^2 + \eta^2/\lambda \qquad (T20)$$

One may refer to Appendix L for a systematic study of different finite solutions among which the two above mentioned ones found in the literature of “doubly special relativity” through Refs.[7,8]. One of the main interests of the present approach is that it is rooted in dynamics and directly associated with the principle of relativity contrary to many works based on empirical generalisations as the ones mentioned above to which one may add another one [36] related to finiteness but with respect to acceleration and not energy.

Comment on some mathematically oriented approaches associated with anisotropy and differential aging reversal.

As emphasized all over this work, the present Leibnizian approach of dynamics differs from conventional approaches in the fact that dynamics precedes kinematics and determines it. As shown elsewhere in this work, the determination may be obtained following the line of thought developed by Taylor and Wheeler on invariants. Here, instead of deducing dynamics $(E^2/c^2 - p^2 = m^2c^2) \Leftrightarrow (c^2M^2 - p^2 = m^2c^2)$ from kinematics $(c^2dt^2 - dx^2 = c^2d\tau^2)$ we deduce kinematics from dynamics, by use of the same procedure that favours the invariants m and $d\tau$ as shown in Ref.[12]. Such a procedure applied to (T11) leads to

$$\{[c^2\mu_\gamma^+ dt - dx]^{1+\gamma/\mu} [c^2\mu_\gamma^- dt + dx]^{1-\gamma/\mu} = c^4\mu_\gamma^+\mu_\gamma^- d\tau^2 \qquad (T21)$$

or equivalently to

$$\{[cdt^+ - dx]^{1+\gamma/\mu} [cdt^- + dx]^{1-\gamma/\mu} = c^2\, d\tau^+\, d\tau^- \qquad (T22)$$

where we have set

$$dt^+ = c\mu_\gamma^+ dt \quad \text{and} \quad dt^- = c\mu_\gamma^- dt \qquad (T23)$$

as well as

$$d\tau^+ = c\mu_\gamma^+ d\tau \quad \text{and} \quad d\tau^- = c\mu_\gamma^- d\tau \qquad (T23)$$

Noting that since we have $\mu_0^+ = \mu_0^- = \mu/2 = [\lambda]^{1/2} = 1/c$ as shown above, it is immediately checked out that when $\gamma = 0$, then one recovers the usual Lorentzian kinematics $c^2dt^2 - dx^2 = d\tau^2$.

Instead of dealing with a double copy of time, one may benefit from the properties associated with conservation laws by recalling that if C and D correspond to conserved entities then, any linear combination aC + bD corresponds also to a conserved entity. With this in mind, one may transform Eq. (T11) as follows:

$$(E^* - c\,p^*)^{1+S}\,(E^* + c\,p^*)^{1-S} = C^2 \qquad \text{(T24)}$$

where we have set

$$S = \gamma/\mu \qquad \text{(T25)}$$

and

$$E^* = (\mu/2)\,E \qquad p^* = p - (\gamma/2)\,E \qquad \text{(T26)}$$

The application of the procedure associated with the one developed by Taylor and Wheeler as to the link between dynamics and kinematics leads to

$$(cdt^* - dx^*)^{1+S}\,(cdt^* + dx^*)^{1-S} = c^2 d\tau^2 \qquad \text{(T27)}$$

or equivalently

$$[(cdt^* - dx^*)^2/(c^2dt^{*2} - dx^{*2})]^S\,[c^2dt^{*2} - dx^{*2}] = c^2 d\tau^2 \qquad \text{(T28)}$$

so that it may be extended to a multidimensional space-time as follows:

$$[(cdt^* - \mathbf{n.dx^*})^2/(cdt^{*2} - \mathbf{dx^{*2}})]^S[c^2dt^{*2} - \mathbf{dx^{*2}}] = c^2 d\tau^2 \qquad \text{(T29)}$$

Let us note that although these structural relations are not clearly associated with specific observed physical phenomena, their existence is not arbitrary but it is based on the principle of dynamical relativity. In addition, one may encounter such structures in the literature among many others as shown in [33-35] and [37], but the justification of such relations is not based on the principle of relativity but on empirical ideas or extended geometries, where one does not limit oneself to quadratic forms. Such more or less founded solutions are sometimes considered as a manifestation of the limits of the principle of relativity. This is mainly due to the fact that the principle of relativity is restricted to space-time theories, intimately linked to the three well-known geometries: Euclidean; Lobatchevskian (hyperbolic) and Riemannian ones. If other geometries provide a wider framework one should recognize that the numerous possible solutions do not lead anymore to a predictive approach because of the lack of sufficiently constrained structures as the ones provided by the present relativity principle rooted in dynamics through conservation laws. In spite of the obtained different solutions, these may be classified in an ordered way and lead to well-determined dynamical systems whose number remains much less numerous than those provided by the use of either imagination or intuition or pure geometrical considerations. Let us finally note that the consideration of functions which are not even for energy does not mean that one deals necessarily with anisotropy, (as in Ref.[34]) since the reflection operation, responsible of this fact, could be associated with time and not with space leading to other physical interpretations.

Appendix U

A Heuristic process of discovery and "principle of fragility of good things".

The controversies about the first person to have discovered a new and original physical idea are partly due to the presence of such an idea in a diluted manner in the atmosphere of the physical community. Many discoverers may simultaneously think of such a change, before fixing it in a rational manner. Putting aside the feeling of nationalism – each nation defends its discoverer as shown by the names of some mathematical theorems and physical laws – an idea is like a seed; in order to grow, it should be planted in a fertile ground. This means that, to develop and be fruitful, the pertinence, originality and potentiality of an idea should be recognized by the community otherwise it dies soon after its birth. Many ideas may be rejected for different reasons – not always rational – especially by use of authority arguments. Some ideas impose themselves with a quasi-necessity but the main problem is to know what to do with them and how to handle them in a fruitful way. We shall consider two examples that played a major role in the development of the present work. The first of these is related to dynamics, while the second is associated with the harmonic oscillator.

(i) Energy, impulse and mass in dynamics.

When dealing with Newtonian dynamics: $E = 1/2mv^2 + U$, $p = mv$, $M = m$; every student who knows elementary differential calculus, realizes the following order $p = dE/dv$ and $M = d^2E/dv^2$ so that the knowledge of E is sufficient to deduce the two remaining quantities. This harmony constitutes a sort of rationality and unity, since the whole information is fixed in a unique entity: energy. One may classify the students in two categories: those who know basic physics and those who do not (ignorant of the Lagrange-Hamilton formalism). The first category knows that impulse derives from the Lagrangian and not from energy ($p = dL/dv$ instead of dE/dv). The local character of Newtonian physics explains why two qualitatively different entities may coincide in some particular situations. For this category, the problem is solved in this way especially that when passing to Einstein's physics, the above-mentioned harmony between E, p and M apparently collapses since the process of derivation is not operational anymore. However, those who know basic physics (through Lagrange-Hamilton formalism) are not conscious of the fact that this kind of physics does not constitute all what can be said on motion. Indeed, the rationality of the so-called basic physics is mainly founded on the notion of velocity, itself constructed by use of the space and the time concepts. Here the ignorant of what is supposed to be fundamental remains open to other possibilities such as the one developed by the "emergent rationality" where motion is not accounted for through the velocity concept. **Those who know basic physics may be blinded by their knowledge when considered in an absolute way.**

Before passing to the second example let us note that mathematically oriented students may find that the structure of Einstein's dynamics: $E^2 - p^2 = m^2$ (hyperbolic curve) is to be parameterised through $E = m \cosh w$ and $p = m \sinh w$ (w: rapidity) rather than the usual well-known expressions associated with the velocity $v = dx/dt$. The usual ones are somehow artificial from a purely mathematical standpoint since they do not correspond to the natural parameterisation of a hyperbolic curve. Let us also notice that the natural parameterisation verifies $p = dE/dw$ so that the above mentioned harmony is recovered if one adopts another point of view on motion. These remarks will constitute a bridge to the next example associated with the oscillator and to my discovery of a possible new rationality and its extension to novel contexts.

(ii) The harmonic oscillator: Analogy between trigonometric and hyperbolic functions.

This second example played a major role at two levels in my work. It firstly allowed me to establish certain new relations between the conventional approach of dynamics called in this work "usual rationality" and the one associated with the new available method developed recently (that we call "emergent rationality"). Secondly, reasoning by analogy allowed me to obtain a generalization of the hyperbolic dynamical framework, following the line of thought developed in the simple passage from the undamped to the damped oscillator. I recently realized that the solutions I had obtained long ago in a purely analogical way, unarticulated to any physical experiment or geometrical structure, have their counterpart in physics and geometry. The logic behind this construction runs as follows:

It is immediate to note the structural analogy between the differential equation: $y'' + y = 0$ associated with the harmonic oscillator, whose integration leads to $y^2 + y'^2 = C$ and its hyperbolic counterpart: $y'' - y = 0$, $y^2 - y'^2 = C$. If this second structure does not correspond to any oscillatory phenomenon it can nevertheless be associated with Einstein's dynamics: $E^2 - E'^2 = E^2 - p^2 = m^2$ because of its hyperbolic character, where it is readily checked that the impulse derives from energy. In spite of the fact that the parameter with respect to which one derives ($E' = dE/dw$) does not coincide with the velocity concept v, the latter seemed to me worthy of consideration from a structural point of view. Here lies my first interest in the rapidity parameter at an epoch I did not suspect the importance it will play in physics and in my personal understanding of dynamical relativity. Like all young students and researchers in a field that they do not possess sufficiently, my ignorance of the importance of $E^2 - p^2 = m^2$ to physics in general, led me to propose a possible generalization by analogy to the well-known one that leads to the damped oscillator where the solution of the form $y = \exp(Sx) \cos x$ becomes in the hyperbolic case $E = \exp(Sw) \cosh w$. In particular, on completing the structure of dynamics as follows: $p = \exp(Sw) \sinh w$ one gets a dynamics of the following form: $(E - p)^{1+S} (E + p)^{1-S} = [(E - p)/(E + p)]^S [E^2 - p^2] = m^2$ recovering $E^2 - p^2 = m^2$ for $S = 0$. Another major point that played a central role in my discovery of an associated kinematics to the above derived dynamics is due to a method developed by Taylor and Wheeler in Ref.[12]. This method consists in a direct association of time to energy and space to impulse, starting from $dt^2 - dx^2 = d\tau^2$ and leading to $E^2 - p^2 = m^2$. (The method is given elsewhere in this work). One thing that one needs to know is that it operates in the two directions: from kinematics to dynamics and from dynamics to kinematics. Unlike the work of Taylor and Wheeler [12] rooted in kinematics and associated with the restricted framework of Einstein's dynamics where $S = 0$, here dynamics precedes kinematics and S differs from zero. In spite of these differences, the method still applies and one derives the following kinematical framework: $(dt - dx)^{1+S} (dt + dx)^{1-S} = d\tau^2$, where the metric enters neither in the framework of hyperbolic geometry $\boldsymbol{\eta}\mathbf{dxdx}$ nor in the framework of the more general Riemannian metric $\mathbf{g(x)dxdx}$. This fact is important in the sense that if one needs to extend the usual well-known geometries (which are at the basis of our presently available theories and which belong to quadratic forms), then one should replace the quadratic form associated with the hyperbolic geometry or Riemann's one by some other form. But, in the same manner as Huygens quadratic function mv^2 (which is also that of Newton) maybe replaced by an infinite number of other functions that lead locally to Huygens dynamics, the quadratic Riemannian metric may be replaced by an infinite number of other more general forms that lead locally to the initial metric. This infinite multiplicity prevents one from getting a predictive framework. Here lies the interest of the present Leibnizian methodology that allows the selection of generalized metrics without being lost into arbitrariness.

Principle of fragility of good things : Leibnizian and Einsteinian relativities.

A thing to happen needs not only favourable circumstances or an open scientific atmosphere, but it should be correctly articulated to different ideas that do not contradict each others. Note that Leibniz was in favour of a finite velocity while his defence of Huygens dynamics contradicts this finiteness requirement (admitting infinite values). However, his defence of Huygens against Descartes' system was justified locally, in the vicinity of the origin (Descartes failed to articulate the state of rest to the state of motion). Moreover, Leibniz knew very well that the parabolic world of Huygens may be local: he asserted at different occasions that a parabola may be regarded as the limit of an ellipse where one of its foci is cast to infinity. We know today that the passage from an infinite velocity to a finite one can correspond to the passage from a parabola to an ellipse since in replacing the parabolic Lagrangian by an elliptic one the structure of Einstein's dynamics (with its finite velocity) is directly obtained. Moreover, the term "Action" at the basis of the principle of least action (Lagrange-Hamilton formalism) was baptized thus by Leibniz who actively contributed to the development of what became later on the Lagrange-Hamilton formalism. It is then tempting to see in Leibniz a forerunner of Einstein's dynamics. A number of authors make such claims, but a deeper analysis of the situation shows two different arguments opposed to this interpretation. The first of these is that Leibniz privileged dynamics to kinematics; the second is that the principle of "least action" (associated wrongly with the "best world" as shown elsewhere in this work) operates on one point of view, while Leibniz was looking for an approach of motion including an infinite multiplicity of points of view. This shows that the methodology associated with the "principle of least action" belongs to only one perspective. On claiming that Leibniz was a forerunner of Einstein, **one reduces his methodology to a unique perspective which is an impoverishment of his conceptual framework**. To be a forerunner of Einstein's dynamics Leibniz's procedure should fill four requirements: the idea of finiteness of motion, the elliptical form as a solution, the method to operate the articulation (principle of least action), and the consideration of space and time as a basic substrate on which the principle of least action operates. Only the first two were emphasized by Leibniz. Although the contribution of Leibniz to what will be called later on "the variation principles" is recognized, the implementation of these principles in space and time is problematic because the concepts of space and time are relegated to the second plan in Leibniz's methodology (as shown all along this work). In brief, what we mean by the "principle of fragility of good things" is that a good thing, to happen, requires the convergence of different elements (here four, where only two are emphasized by Leibniz and regarded as essential while the other two remain secondary). This does not mean that the incapacity of Leibniz to obtain Einstein's relativity is a bad thing: it means, on the contrary, that Leibniz's approach is, in some regard, deeper than that of Einstein, which remains glued to the unique point of view associated with the velocity concept.

Let us also emphasize that unlike the "usual originality" of different revolutionary scientists, such as Einstein or Heisenberg, the originality of Leibniz does not lie in his defence of absolutely new ideas never thought of before. Leibniz champions the defence of **continuity rather than rupture**. He adheres to the **principle of continuity** in its **philosophical** and **mathematical** acceptation of the term. He certainly, proposed genius interpretations of old ideas, but he never accepted the Cartesian mechanistic idea of a total new conception of the world. There is behind this conciliatory attitude (between distant disciplines and apparently contradictory assertions) certain wisdom, based on the **condemnation of violence**. As pointed out by C. Frémont, the theological question of "trans-substantiation", at the centre of numerous letters with Des Bosses, is not superficial as usually considered but an essential one. This question sheds light on the **philosophical and physical** problem of substance, not only the **theological** one. This led the author to read the letters addressed to Des Bosses independently of the circumstances that led to them, but in an intrinsic way, concentrating on their internal logic, that lies beyond the specific problem of trans-substantiation, God's incarnation and hypostatic union understood to be the problem of how there can be a union of the divine and human natures of Christ in one substance. This theological problem of hypostatic union seems an especially appropriate context for a discussion of how the active (mind) and passive (body) principles are to be related in a corporeal substance. These considerations, favoured by Leibniz are rooted in the "principle of analogy" that transcends the different contexts and that shows the **possible existence of the same internal logic, hidden behind contexts semantically remote from each other**. Such analogies, widely used in physics and constituting a bridge between different physical phenomena (mechanics and electricity for example) are never considered, explicitly, between physics and metaphysics. A recognition of such a possibility goes against the usual physical paradigm anchored in Kant's absolute distinction between the "physical" and the "metaphysical". The two thousand years of metaphysical thinking is totally rejected of the scientific paradigm, under the pretext that this kind of logic turned out to be of no help for scientific discovery. This proves its sterility. However one should keep in mind that **the absence of a proof is not a proof of an absence.** The present Leibnizian framework shows that the absence of a tree-like structure associated with motion is not a logical impossibility. It corresponds simply to what physics refused to examine by admitting only what is actually measurable and rejecting any potentiality possibly actualized in the future. Leibniz's epistemology shows us that unlike Kant's belief in the validity of the Newtonian framework, there is much more to say than what the space-time physics of Newton and Einstein assert about the question of motion.

Appendix V

Extension to broken parity of Klein-Gordon and Schrödinger equations.
It is quite well-known that the structure of dynamics may be associated with the structure of Klein-Gordon differential equation, Schrödinger equation, etc. In this Appendix, we shall perform an extension to frameworks where parity or symmetry reflection is broken, by use of the following correspondence laws:

$$E \to i\,\partial/\partial t \qquad p \to -\,i\,\partial/\partial x \qquad \text{(natural units)} \tag{V1}$$

Such a correspondence leads to interpret dynamics as a dispersion relation of some differential equation.

(i) Extended Klein Gordon equation.

On applying the above-mentioned procedure to the following extended dynamical framework

$$(E - p)^{1+S} (E + p)^{1-S} = m^2 \tag{V2}$$

one is left with

$$[(\partial/\partial t + \partial/\partial x)^{1+S} (\partial/\partial t - \partial/\partial x)^{1-S}]\, \psi = -\, m^2\, \psi \tag{V3}$$

where it is readily checked that when $S = 0$, one gets the well-known Klein-Gordon equation.

(ii) Extended Schrödinger equation.

In order to apply the procedure to Schrödinger equation, one should recall that the latter is not related to Einstein's dynamics but only to its local counterpart associated with small impulses and energies. This fact leads us to proceed in two steps, one firstly performs a localisation procedure then applies the correspondence laws.
The localisation procedure follows the line of thought developed in the passage from Einstein's dynamics to Newton's one. Starting from (V2) or equivalently from

$$m^2 = (E^2 - p^2)(\{E - p\}/\{E + p\})^S \tag{V4}$$

one may write

$$E^2 - m^2 = [E^2 - (E^2 - p^2)(\{E - p\}/\{E + p\})^S] \tag{V5}$$

so that on defining kinetic energy by setting

$$T = E - m \tag{V6}$$

as usual one gets then

$$T = [E^2 - (E^2 - p^2)(\{E - p\}/\{E + p\})^S]/[E + m] \qquad (V7)$$

On taking the limit of small impulses ($p \to 0$) and kinetic energy ($T = E - m \to 0$), one is left with

$$T = [p^2/2m]\,[1 - 2Sp/m] + Sp \qquad (V8)$$

It is readily checked that for S = 0 one recovers Newtonian dynamics: $T = p^2/2m$. We have neglected here the terms such that $(p/m)^2 \ll 1$, but we kept those of first order in p/m.

On recalling that, in the presence of a potential contribution U the total energy Ê corresponds to the addition of kinetic energy T to the potential one U as follows

$$\hat{E} = T + U \qquad (V9)$$

The application of the usual already mentioned correspondence relations applied to the couple (Ê,p) leads to

$$i\partial\psi/\partial t = [1/2m][1 - \{2iS/m\}\partial/\partial x]\,\partial^2\psi/\partial x^2 - iS\partial\psi/\partial x + U\psi \qquad (V10)$$

In the absence of S, one recovers the usual Schrödinger equation

$$i\partial\psi/\partial t = [1/2m]\,\partial^2\psi/\partial x^2 + U\psi \qquad (V11)$$

Appendix W

Articulation of the couple "essence–existence" or "substance–monads" through a "substantial link" at the basis of Leibniz's dynamics.

Leibniz follows Aristotle more than Plato in dynamics: Plato focuses the attention on "ideas" while Aristotle concentrates in the first place on "**concrete individuals**", that Leibniz calls "**monads**". These monads constitute different **subjective manifestations** of an "**essential being**" which is **objective** since it is associated with a unique "**main object**" that Aristotle, and Leibniz after him, call a "**substance**" or an "**essence**". In dynamics, the substance is associated with the unique entity without which dynamics looses its very existence. This unique entity (substance or **essence**) is reflected through the **correlation between the two conserved quantities**: relation of energy to impulse. This fact constitutes a first step towards what is known by Leibniz as the "relational framework". Many scholars put emphasis on the fact that substance (in Leibniz's philosophy) is not a "thing" but a **relation** but this relational character of substance leads them to a "mathematical view" of physics. (Leibniz is known as a genius mathematician but not as a physicist in the classical acceptation of the term). If Leibniz does not conceive physics without mathematics, it is because the emphasis is put on **relations and procedures** rather than on things: Democritus "atoms" or Newtonian "particles" constitute a **naïve conception of "substance**". Mathematics remains a precious tool that helps the effective actualisation of his relational view of "substance". This relational view does not consist in establishing only a relation between the above mentioned conserved entities, but also (and above all) in constituting an infinite multiplicity of relations between "substance" and its unlimited number of "monads" (each of them representing a particular manifestation or a modality of existence). Here lies the basic difference between the usual "relational view" of mathematicians and the one proposed by Leibniz: there is a violent contrast between the Aristotelico-Leibnizian view of physics and the Parmenido-Newtonian vision that misses the existence of an infinite multiplicity of "concrete individuals or monads". The Parmenido-Newtonian vision put emphasis on one "concrete individual" or one special "monad" if one uses the Aristotelico-Leibnizian terminology. Parmenides put the first cornerstone of philosophy by his introduction of the "one" or the "essential being" that Aristotle called "substance". But, he did not propose any articulation between the "one" and the "many" or equivalently between "substance" and its "modalities of existence". One of the main characteristic features of Aristotle's ontology is concentrated in his basic assertion according to which **"substance" can be said in a multiple ways and through different perceptions and representations,** each of which associated with one modality of existence. Leibniz associates each way, perception or representation to one "monad" that reveals some features and facets of "substance". Each "monad" constitutes one modality of existence, while substance that lies behind these modalities, needs them to be revealed. In brief, one may say that Leibniz's ontology, inherited from Aristotle's, is at the same time "many" and "one". It is "many" in its search for an art of distinctions between multiplicities of modalities of existence, but it is also "one" in its concentration on a unique centre constituting "essence". One of the main reasons for which physics is so difficult to deal with (from a conceptual standpoint) is that this discipline was erected on an empirical basis confusing **essence** with a **unique modality of existence,** no distinction is made between the two realms : essence and existence, "substance" and "monads", "one" and "many" or "objective entity" and "subjective manifestations". Worse, physics and particularly mechanics (considered as the queen of science which served as a model for physics and which constitutes its foundation through Noether's theorem) begins systematically by defining existence without regard to "essence". This lack of basic distinctions between "substance" and "modalities of existence" is responsible for the misunderstanding of Leibniz's vision and its confinement to the realm of mathematics. At different occasions (and particularly in his correspondence with Huygens), Leibniz emphasized that the above-mentioned metaphysical and ontological considerations (that need the construction of an adequate formal language) aimed ultimately at the foundation of a science on a firm ground. But Leibniz was preaching in the desert of the Newtonian world.

Newton and his disciples, like ants in the desert, believed that the world is constituted of an infinity of grains of sand (Newtonian point particles) placed in an empty infinite and absolute space. **The main feature that distinguishes Newton from Leibniz is that Newton centres his effort on homogeneity and emptiness, while Leibniz anchors his vision in the variety of things and their correlation to each other, in order to form a true unity so as to a unity by aggregation**. The difficulty in dealing with Leibniz methodology is due to the fact that the language of physics (inherited from the Newtonian system of the world and represented mathematically by specific correspondences), does not permit a rational discourse on Leibnizian physics, for it is too narrow. In spite of the passage from the Newtonian universe to the Einsteinian one, the same language and methods are kept invariant. Only the articulation of things differs significantly. To fix ideas, let us note that one needs three variable numbers to deal with dynamics in a one dimensional framework. Two of these represent the conserved entities while the third one accounts for motion. This is true in Newton's world as well as in Einstein's one. No net distinction is made of what amounts to "essence" and what amounts to "existence". Yet, such a distinction is essential, since "essence" is "one" while "existence" is "many". On confusing the "one" with the "many" it is not surprising anymore to find out that at a certain level, physics becomes contradictory and difficult to grasp. When physicists realized (at the end of the 20th century) that the velocity concept associated with motion was not operational anymore, they were obliged to change of point of view on motion. This fact is logically possible simply because the account for motion through the velocity concept does not constitute a necessity, unlike energy or impulse the absence of which leads to the absence of dynamics itself. In brief, one may say that energy and impulse are the basic constitutive elements of dynamics unlike the velocity. In Leibniz's terminology the relation between energy and impulse belongs to "essence" while the relations of energy and impulse to the velocity belong to "existence". This is the reason why one may continue to deal with dynamics without the use of the velocity concept replacing it by another "existential" entity. If velocity belongs to "essence" as usually believed, the above mentioned replacement could not be done. Modern physics was finally obliged to distinguish between "essence" and "existence", in facts, even if this philosophical terminology continues to be ignored by physicists. The physicists pass from one "existential entity" to the other only when they are obliged to do so. There is no theory that tells the physicists if these "existential entities" are possibly articulated to each other, and if, as we affirm it, the answer is positive, what kind of articulation is at work. This lack of knowledge concerning the possible correlations between the different "existential entities" is a direct consequence of the too narrow mathematical language used to account for the structure of dynamics. There is no place for any multiplicity of points of view on motion in the very structure of dynamics.

Each time one is faced with an existential difficulty proper to motion, one is condemned to propose another framework. This is exactly what happened with the development of what we have called the "emergent rationality". Unlike the "usual rationality" provided by the Lagrange-Hamilton formalism, the "emergent rationality" uses a completely different methodology, founded on "group theoretical methods", which differ radically from the Lagrange-Hamilton formalism. All these considerations and developments constitute a concrete (although partial) manifestation of the possible validity of the Leibnizian monadic thesis based on the necessary distinction between "essence" and "existence", or more precisely between "substance" and its various "modalities of existence". The above mentioned two "rationalities": the "usual" and "emergent" ones indicate that the physical community tends to follow what Leibniz proposed long ago without recognizing yet this fact. [One should add a third rationality due to Taylor and Wheeler even if it is less systematized as the other two]. Like Mr Jourdain making prose without knowing it, physics is following the path paved by the Aristotelico-Leibnizian distinctions of "essence" and "existence", "one" and "many" or more precisely by "substance" and "monads". However, this fact is still unrecognized for two reasons: one extrinsic and another intrinsic. The extrinsic reason is that physics has no special regard to its history: by its very nature it aims at the future and examines the last novelties. Leibniz's epoch appears as the Stone Age for modern physics. The intrinsic reason is that even those who deal with general subjects (like epistemologists and philosophers of science) still live in a Kantian paradigm that rejects the Aristotelico-Leibnizian metaphysics, and particularly what Leibniz calls a "substantial link", needed to articulate substance to monads (that correspond to different modalities of existence on the same essence). In addition, the Leibnizian "monads" constitute an **infinite** number of perspectives on an "essential reality" (here dynamics) and this infinite character seems to be a major element since the Leibnizian perspectives are not arbitrary but organized according to some law (substantial link) that repeats itself indefinitely, a recurrent law that Leibniz illustrated by the ratio of a geometrical progression. Leibniz extends the well-known Aristotelian analogical thinking (a/b = c/d) to infinity, each term being not a simple word (as in Aristotle's assertion "old age is the evening of life": old age/life = evening/day) but is a monad that accounts for one point of view on substance mathematically expressed by a function ($U_{n+1}/U_n = U_n/U_{n-1} = \ldots = r$) where the different terms U_k (monads) are functions of some variable x. It is this **infinite repetitive procedure** governed by the ratio (r) of the progression which ensures the unity of substance (and which depends automatically on the variable x because of the U_k dependence) which **constitutes the basic Leibnizian intuition** as to how the monads can be related to substance. Here, one immediately notices the importance of the infinite number of monads without which no general order is possible. Here lies the main difference with the usual conception of physical formulations and the great distance that separates Leibniz's methodology from the usual adopted procedures in so far as the construction of physical methods are concerned. As long as no physical formulation satisfying the above considerations is taken into account (with an infinite multiplicity of points of view on motion), one cannot say if Leibniz is right or wrong. However, if it is impossible to prove that Leibniz is wrong, it is possible to prove that he is right in dealing with dynamics the way he does, and showing that **the "substantial link" is a reality, and not a "ghost"** as usually believed. It is sufficient to show that, when handled appropriately and when the concepts of "substance" and "monads" are dealt with rationally through an appropriate language, one gets a dynamical framework linkable to presently available dynamical formulations. Obviously, such a research program requires the use of a sort of **"inclusive logic" capable to deal simultaneously with an infinite number of monadic entities.** Such a requirement leads automatically to the necessity of constructing a specific formalism, rich enough to encompass the evoked infinite multiplicity of "monads" and their hidden order. This order is impossible to detect by the available methodologies since each favours one perspective or one "monad" from the start. The use of the Lagrange-Hamilton formalism selects the velocity notion at the expense of other degrees of freedom (a "monad" for Leibniz or a "concrete individual" for Aristotle). The use of group theoretical methods leads to the selection of another modality of existence known as the rapidity relegating other perspectives to a second plan. [The use of still a third formulation such as the one favoured by Taylor and Wheeler, based on metrical geometry associated with invariants, constitutes a different point of view as compared to the two preceding methodologies. The basic idea associated with this third methodology – that has not been developed systematically in dynamics as the two other ones – consists in assuming that the **real physical entities** are those which correspond to invariants, so that among the different properties encountered in a theoretical framework, it is natural to favour the invariant ones considering them as the cornerstone of the general architecture of the proposed formulation].

One of the big surprises relative to this program is that the use of the differential calculus along the line of thought developed by Leibniz leads to the possibility of encompassing the different results provided by the three methodologies into a unified framework, so that the **epistemological disorder** resulting from the multiplicity of the proposed methods may be absorbed by the discovery of a **hidden ontological order** based precisely on the fruitful Aristotelico-Leibnizian distinction between “Substance” and its “modalities of existence” or “monadic perspectives”. In addition, the link between the two objective and subjective realms is achieved with the exhibition of what Leibniz calls a “substantial link” usually considered to be a purely metaphysical entity with no relevance to the physical world.

Let us note that in each formulation one may encounter invariant properties (Taylor and Wheeler), quantities that verify group structures (Lévy-Leblond and Comte) and others that correspond to a minimum (Lagrange and Hamilton). However, the important thing is that these different properties are not of equal importance. Each methodology centres its investigation on one of them considering it as the central element to begin with. On favouring invariants, one is led to the privilege of “invariant time” also called “proper time”, so that motion is accounted for through the so-called “proper velocity”. On favouring group theory, the attention is then drawn on a theorem that asserts the existence of an additive parameter that will account for motion known as rapidity. On favouring the principle of least action developed by Lagrange and Hamilton one is automatically led to the velocity concept as the “fundamental” entity associated with motion. This methodology is the one widely recognized and considered to be the most basic one, since it corresponds to the first rational approach of motion, on which a great number of investigations has been proposed, among which the famous Noether’s theorem and gauge theories using the Lagrange-Hamilton formalism as the basic substrate through which they operate. It is the exclusivity of this well-known methodology as to the foundation of dynamics that was recently put into question by some authors among which C. Comte. The multiplicity of different possibilities clearly suggests that the term “fundamental” is not appropriate here since what appears to be fundamental and primary in one method turns out to be secondary in another one. Each method constitutes one point of view so that the term “best” does not fit, as sometimes asserted, to the characterization of any one of the different methods. In Leibniz’s methodology, the “best” operates at the level of “objective possible worlds” but not on the “subjective points of view”. In other words, the “best” appears in dealing with “essence” and not with “modalities of existence”.

Appendix X

On the possible origin of the multiplicity of points of view in Leibniz’s methodology.
I wondered for a long time at the **possible origin** of the fruitful Leibnizian methodology associated with the multiplicity of points of view or perspectives, and I would like to propose two complementary answers. Both of them are linked to the importance of conics in Leibniz mathematical studies. The way Leibniz looked at conics – corresponding to the structure of Einstein’s dynamics (hyperbolic) as well as the Newtonian one (parabolic)– in relation to the multiplicity of points of view is clearly accounted for in the study of the catenary’s curve as shown in Appendix E. In this regard, let us recall that, after having thought that the catenary’s curve was parabolic as asserted by Galileo, Bernouilli, Huygens and Leibniz discovered each in his own way and using a different parameterization, that the form of this curve is associated with the hyperbolic structure rather than with the parabolic one. In addition, Leibniz knew that, locally, the three different parameterizations become fused into one parabolic form. From a structural point of view, we see here, concentrated in a unique problem, the three elements associated with

the multiplicity of points of view on the hyperbola and their fusion into a unique parabolic form. This can be geometrically grasped through a simple intuition: a tree-like structure (here the three representations of the hyperbola) may be locally fused into a unique parabolic form (the trunk).
If this explains the importance that Leibniz attached to the multiplicity of points of view, this does not explain the requirement of an infinite multiplicity of such points of view. In his "monadology", Leibniz illustrates the multiplicity of points of view by the image of a town that one may look at from different locations. Obviously, the locations from which one looks at a thing are effectively infinite, but this example is too simplistic to be of any structural relevance. There is another structural example than the catenary's curve which pledges in favour of such an infinite ordered multiplicity. The latter is anchored in the Leibnizian definition of a straight line, which is partially Aristotelian. The structure of the Aristotelian analogy ($a/b = c/d$) may be used to define a straight line provided that one of the ratios correspond to infinitesimal entities instead of finite ones: $dy/dx = y/x$. According to Leibniz, a straight line is characterized by the fact that the ratio associated with global measurements (segments y and x) is equal to the ratio of local ones (infinitesimal or vanishing segments dy and dx). Starting from this property and multiplying the infinitesimal ratios, one may get an infinite number of straight lines that differ only by the value of the integration constant C so that if one fixes it to unity ($C = 1$), one is left with an infinite number of fused straight lines.

The remarkable point here is that the simplest deformation of this configuration leads to a tree-like structure with an infinite number of branches locally fused together (as shown in Appendix H). The reading of different (mathematically oriented) Leibnizian texts suggested to me that Leibniz could have done such a calculation but I ignore if this has been effectively done. Whatever the answer to this specific question, Leibniz was able to perform the following relation associated with the hyperbolic system $x_\mu = \int dX/(1+X^2)^{\mu/2}$ where x_2 and x_1 correspond respectively to the parameters used in the study of the catenary's curve by Huygens and by himself. Let us recall that in the 17th century the definitions of Arctan and of Argsinh were not yet classified by their properties as done today but their knowledge was intimately linked to the way they were generated through their development in series by use of $1/[1 + X^2]$ and $1/[1 + X^2]^{1/2}$ which belong to the following family of curves :$1/(1+X^2)^{\mu/2}$. Leibniz was sensitive to these curves: at different occasions he used to replace the particular form $Y(X)$, $[Y(X)]^2$, …. by a unique but nevertheless infinite multiple extended one $[Y(X)]^n$. According to Leibniz, such an extension may be very useful for investigations and for the discovery of novel perspectives possessing remarkable properties. This also leads to **a true unity out of a scattered multiplicity**. Such procedures lead to a gain at two different levels. One gains in **revealing new elements** absent from the initial data, and in **discovering a hidden ordered unity** that governs the initial entities placed in a disordered way and known only partially.

It is clear that the scattered particular forms: $u = A X$, $v = A X/(1+X^2)^{1/2}$, $w = A \operatorname{arcsinh}X$ and $x = A \arctan X$ lack a true unity while their inclusion as particular manifestations of the following ordered relation $x_\mu = A\int dX/(1+X^2)^{\mu/2}$ reveals their possible origin and hidden order. There is no doubt that Leibniz was in search of such a unity in the study of natural phenomena since these considerations are met at different occasions in his writings. In addition to these general considerations, I suspect that the example of the catenary's curve as his general study of conics through developments in series played a major role as to the importance that Leibniz devoted to the idea of points of view on a given reality. The support of such an idea on a more concrete ground deserves to be deepened through further historical and philosophical studies concerning Leibniz approach of natural phenomena.

Appendix Y

On three complementary ideas behind the dynamical relativity principle associated with relativity, identity of indiscernibles and plenitude.

In the main text, we have merged three different ideas into one appellation, in order not to multiply the different denominations (we have already made a number of distinctions that needed the introduction of new appellations, such as trans-subjectivity and inter-subjectivity). However, one gains in understanding if one separates the so-called principle of dynamical relativity into three parts. The first is linked to the mechanism of **translation** expressed through successive unlimited derivations each leading to a new conserved entity. The second consists in imposing a constraint so that one gets two and only two conservation laws, the others being indiscernible **(principle of identity of indiscernibles).** The third is introduced to account for an infinite multiplicity of points of view on motion **(principle of plenitude)** where translation is expressed through many different ways. The first requirement follows the line of thought developed by Huygens who uses the idea of translation: v + V starting from the "living force" and deducing the "quantity of progress" or impulse mv by use of $m(v+V)^2$ and its combination with mv^2 as shown in the main text and in Appendix A. The second requirement is closer to Leibniz's spirit where one does not accept the mv^2 as a given entity justified experimentally, but one looks for a rational justification obtained by the enunciation of a principle: the **"principle of identity of indiscernibles".** After recalling that Huygens's combinatorial procedure is equivalent to another procedure using Leibniz infinitesimals and making of the derivative a generator of conservation laws, one shows that Huygens dynamics, reviewed by Leibniz's systematic procedure leads to a **complete rationality** where mv^2 is not related to any experimental evidence anymore but deduced from a principle. It is obtained by imposing a constraint (principle of identity of indiscernibles) on the second order derivative, in such a way that one gets two and only two conserved entities. In the Newtonian framework (compatible with that of Huygens) the indiscernible character is due to the vanishing of all the derivatives of orders higher than two ($d^2E/dw^2 = dp/dw = \text{Const.}$) while in the Einsteinian one, the higher order derivatives coincide with energy and impulse alternatively ($d^2E/dw^2 = dp/dw = E/c^2$) so that no new conservation laws are obtained in spite of the unlimited number of derivatives. As to the idea of **plenitude** that ensures the existence of an infinite number of points of view on motion, it is given through **a scale recurrent law** that allows tounfold the different points of view one after the other in an ordered manner. These three considerations are expressed in a compact manner through the following expression

$$\mathbf{M_i = d_{\mu i}{}^2E/dv_{\mu i}{}^2 = (E/E_0)^{(a-\mu)i}\, d/dv_{\mu i}\,[(E/E_0)^{(a-\mu)i} dE/dv_{\mu i}] = m(E/E_0)^i} \qquad \textbf{(Y1)}$$

derived in the main text through Eq.(18), linked to Leibniz's methodology that includes an a priori, doubly infinite multiplicity: inclusive points of view and exclusive possible worlds (indicated respectively by the Greek and Latin indices μ and i). The **indiscernible character** is produced by the constraint imposed in the right hand side of (Y1) through the following **simple**

scale law: $\mathbf{S_i = m(E/E_0)^i}$ **,** the Latin index i is fixed for each possible dynamics. The **plenitude** character is ensured by the following **recurrent scale law :** $\mathbf{R_{\mu i} = (E/E_0)^{(a-\mu)i}}$ where the Greek index μ is not fixed but takes an, a priori, infinite number of values (a, a+1, a+2,….). One recalls that among the possible dynamics (Leibnizian possible worlds), the two admissible ones associated with Newton and Einstein correspond to

i = 0 (Newton) i = 1 (Einstein) (Y2)

The indiscernible character is revealed through vanishing quantities in the first parabolic Newtonian dynamical case, and through an infinite reproduction of identical expressions since the couple of conserved entities is recovered indefinitely (identical to itself up to a multiplicative constant which plays no particular role in so far as conservation properties are concerned).

In brief, one may say that what has been called "dynamical relativity principle" (for short) includes three different complementary ideas. The first of these shows how the relativity idea operates in passing from one conserved entity to another, as initiated by Huygens, while the second reveals the kind of constraints to impose in order to obtain a well-determined rational solution through Leibniz's principle of identity of indiscernibles. As to the third one (principle of plenitude), it ensures the existence of an ordered structure leading to an, a priori infinite multiplicity of different points of view on motion. The solution of (Y1) shows that only four points of view turn out to be singular and basic while the others constitute more or less complicated combinations of the four basic ones. The principle of identity of indiscernibles is what is known in philosophy by a "principle of individuation" that selects or individuates one structure among an infinite number of possibilities. **Unlike the Newtonian and Einsteinian individuation mechanisms usually obtained through a reflection on the concepts of space and time, the Leibnizian individuation procedure is obtained in a relational way through the combination of purely dynamical arguments in direct relation to conservation properties.**

Appendix Z

Relation of science to philosophy: Einstein and Bergson.

This Appendix is devoted to lessen the distance between science and philosophy, showing that if science possesses a certain autonomy (as shown since the 17th century and believed by most modern scientists), its foundation may require philosophy for a better understanding of some of its underlying principles. The realm of dynamics – at the basis of physical science where its basic laws belong to the class of "superlaws" (Wigner's denomination followed by many scientists and epistemologists as shown elsewhere in this work) – is significant in this regard. In particular, as long as one keeps thinking in the framework of the "current scientific paradigm", there is no way out of the internal logic associated with this representation of predictive science. Some philosophical issues in direct relation with the problem of the "One" and the "many" (that goes back to antiquity) cannot be tackled with, in the too narrow realms of quantitative physics and mathematics. Before that mathematics shows its supremacy in the area of fundamental physical science, imposing the strict distinction between the "true" and the "false", the door was still open to a certain common sense reasoning that lies beyond such a strict opposition. With the development of modern science initiated by Newton and followed by Lagrange, Hamilton, Einstein and others, the problem of motion became expressible through the **binary logic** of "yes" or "no", "true" or "false". As an example, let us recall that the relation between motion and energy became decidable exclusively in a unique manner and without any way out or subterfuge. When Einstein discussed motion with Bergson, he was embedded in such a logical exclusive framework. The relation between motion and energy was in Einstein's mind unique and definable only through the velocity concept with no other rational possibilities. When Bergson made a distinction between examining a thing **from the inside**, entering into it ("inside unity") and looking at it **from the outside**, turning around it ("outside multiplicity") to observe it under different angles of vision or perspectives, (Chapter VI of Ref.[40] entitled : introduction to metaphysics) these metaphorical distinctions (unusual for a physicist) were considered to be mere superficial assertions. If such a distinction may have some meaning in the history of philosophy and particularly in Aristotle and Leibniz metaphysics that deals with the problem of the "One" and the "many", these qualitative considerations do not apply (for him) to physical science which should be quantitative to be predictive.

For a better understanding of the position of Einstein with respect to Bergson, let us recall that **according to Einstein, Bergson did not understand the relativity theory so that his philosophy is of no relevance to the problem of motion.** This apparently consistent assertion should be split into two distinct parts only the first being admissible. It is established that Bergson had not sufficiently integrated the principle of relativity as discussed by Einstein. Thus, Einstein was right in asserting that Bergson did not understand his theory of motion, fruitful and superior to that of Newton. However, as emphasized all along this work, one should avoid confusing "**fruitfulness**" with "**truthfulness**": what Einstein said about motion is not necessarily the last word. Moreover, the fact that Bergson did not grasp Einstein's relativity does not imply that Bergson's vision of motion is to be wholly rejected and that the distinction between an "**inside unity**" and an "**outside multiplicity**" is to be reduced to mere words with no relevance to the problem of motion. If the physicists follow Einstein in condemning Bergson's philosophical assumptions, this is mainly due to the underlying contingent (but unrecognized) philosophy of physics. This philosophy anchors its methodology in a strict and narrow framework opposing what is "true" to what is "false" in an absolute way and with no intermediate possibilities. In Ref.[40] Bergson proposed a subtle analysis and gave many significant examples showing that an assertion may be **true or false only under certain conditions**, relative to a specific situation and not being of an absolute necessity. He, thus, joins the Leibnizian multiplicity of subjective points of view in so far as the "outside multiplicity" is concerned. Bergson also underlined the possibility to get beyond or transcend this external subjective multiplicity through intuition reaching thus the core or the "inside unity" out of reach in Einstein's dynamics (as well as in all dynamical alternative approaches). This impossibility is due to the fact that physics deals mainly with particular quantitative analytical models. Thus, motion is accounted for quantitatively: there is no place for any qualitative consideration from the start. In refusing the realm of quality, physics refuses by the same token the possibility of a passage from quality to quantity. It is precisely through such a passage that one is able to reach a quantitative version of the "inside unity", by pushing back the infinite multiplicity of qualitative perspectives through a mechanism of compensation. Having repelled the qualitative features associated with the existing but unspecified points of view on motion (outside multiplicity), one is then facing the well-specified and determined "inside unity". This mechanism of compensation rejecting all the qualitative features associated with the modalities of existence (outside multiplicity) and revealing the "essence" of dynamics (inside unity) through the unique quantitative relation (expressing the two conserved entities energy and impulse) is what was lacking to Einstein's vision. The father of relativity theory was too deeply committed in the conventional scientific enterprise to draw back and take seriously philosophical considerations, yet essential in a constructive discussion with a philosopher like Bergson. In addition, according to his own terms, Einstein was closer to Spinoza's philosophy favouring the realm of necessity to that of degrees of freedom proposed by Leibniz perspectives. **If Einstein was conscious of the distinction to be made between the orders of *necessity* (conserved entities) and of *freedom* (non conserved entities such as velocity, rapidity, celerity etc.) he would have listened more carefully to the subtle Bergsonian analysis, even if this philosopher had not a full understanding of Einstein's relativity of motion.** In Ref.[1], footnotes of page 470, one reads" *Mach, through his Mechanics (which Einstein read avidly), may well have introduced the young Einstein to several typically Huygensian techniques, above all the use of the relativity principle. In particular, Mach gives an account in the Mechanics of Huygens' use of it in his work on collisions….. possible that Einstein was therefore directly influenced by Huygens through his reading of Mach*". If Einstein was sensitive to Huygens, he neither adopted the ideas of Leibniz that extend those of Huygens nor the correlated ideas of Bergson following the same line of thought, in so far as the consideration of perspectives on a reality (here dynamics) is concerned. If Einstein's philosophy was closer to that of Leibniz rather than Spinoza he could have benefited from the subtle analysis of Bergson relative to the views associated with "inside unity" and "outside multiplicity" which correspond in the present work to the "trans-subjective unity" (going beyond all points of view) and the "subjective multiplicity" respectively. These two versions of the principle of dynamical relativity (given in the first part of this work) espouse perfectly the subtle Bergsonian analysis. **It is remarkable that the reject of the intrusion of philosophy in the realm of physics resulted in two missing opportunities for a better understanding of the question of motion. The first missed opportunity occurred in the 17th century through the refusal of Leibniz distinction between "substance" and its "modalities of existence", the second occurred in the 20th century through the rejection of the Bergsonian distinction between an "inside unity" vision and "outside multiplicity" views**. With these considerations, one better understands the provocative assertion of Heidegger: "Science does not think" and this other affirmation of Gödel asserting that the technical development of science is much more privileged than its conceptual foundation.

The main problem of physics is that its empirical character – intimately related to experimental evidence – precedes its rational formalisation, so that it remains embedded in the realm of empiricism and contingency, not distinguishing between what amounts to necessary requirements and what amounts to degrees of freedom. More precisely, when Lagrange and Hamilton formulated Newtonian dynamics by use of functional relations, these did not deal with the essence of what could be meant by motion, but only with one of its modalities of existence. With the advent of Einstein's dynamics the relations between the old Newtonian solution and that of Einstein differ but the concept of motion itself remained the same. This is clearly shown by the following writing $v = dx/dt = dE/dp$ which lies beyond the Newtonian and Einsteinian dynamical frameworks since it is verified by both of them. The difference appears only at a later stage where one completes the structure by assuming $v = p/M$ with $M = m$: const. (Newton) and $M = E/c^2$ (Einstein). In order to see why the usual approach of motion including Einstein's procedure is unable to grasp the philosophical Bergsonian distinction between an "outside multiple vision" and an "inside unique view", let us recall that for Einstein, the relation between motion and conserved entities (energy and impulse) is given uniquely through $v = f(E) = g(p)$ where f and g are the two quantitative well-known functions associated with Einstein's dynamics but whose specification is not needed for the present discussion. One concentrates here on the distinction between the general problem of the "One" and the "many" rather than on any particular specification of one relation or another. **In Einstein's vision which is also the one adopted in the usual teaching of physics, there is no place for distinguishing an "outside multiple vision" from an "inside unique view" since multiplicity is absent and its physical presence means a lack of determination and hence unpredictability**. All this reasoning is based on the fact that conventional physics does not make a net distinction between "essence" and "modalities of existence" so that it is well-adapted to the internal logic of analytical mathematics through which dynamics is expressed. However, if one replaces $v = f(E) = g(p)$ by the following multiple forms $v_\mu = f_\mu(E) = g_\mu(p)$, then one is led to the above-mentioned "external multiple vision". The "internal unique view" may be accounted for through the relation between the conserved entities p and E expressed by $p = g^\mu[f_\mu(E)] = G(E)$. The index μ that represents multiplicity should be eliminated by compensation, otherwise no predictive theory is available. (These considerations have been dealt with explicitly all along this work and do not deserve to be recalled in more detail here). The only thing worthy to note is that there is no contradiction with the simultaneous existence of multiplicity and predictability, provided one distinguishes precisely what amounts to "essence" and what amounts to "modalities of existence" (or in Bergson's terminology, what amounts to an "inside unique vision" and an "outside multiple views"). In brief, one may say that **the subtle idea of harmony in the Leibnizian sense, that goes beyond the crude exclusive logical opposition between the true and the false, is the one to be adopted to understand the fundamental opposition between Einstein and Bergson**. This opposition is not necessarily in favour of Einstein, even if Bergson did not understand some of the subtleties of relativity theory. At a more fundamental level, Einstein's formalism embedded in the binary logic of the true and the false and confined in exclusive methodologies – such as the scalar Lagrange-Hamilton formalism or the vector Newtonian one – prevented him from adopting the distinction between "outside multiple vision" and "inside unique view", essential for a better understanding of the idea of motion without sacrificing the possibility of prediction.

It is worth noting that, if Bergson and Leibniz may differ on some basic philosophical points and particularly on the way they articulate "essence" with "existence", the two philosophers share the necessity to go beyond the crude opposition of what is true and what is false, in favour of a multiplicity of visions that complement each other so that what seems to be confused in one vision becomes clear through a different one. The crude familiar opposition in usual rational physics is extended (by the inclusive logical Leibnizian framework) to a subtler exposition of such an opposition, that remains effective only when associated with some specific conditions. The association of space with time leading to the velocity notion is not only unnecessary as shown in numerous recent works but it is also less natural than rapidity, from a theoretical standpoint as well as from a practical one since it does not apply anymore for very high energies (asymptotical behaviour). As long as the conditions are not necessitated by the very structure of dynamics (conservation laws and relativity requirement) but only by the way one measures a physical phenomenon, one should keep the door open to different other kinds of measurements, each of which constituting one modality of existence or one point of view on motion. Here lies the basic philosophical idea of Leibniz also adopted by Bergson through his distinction between the intrinsic way to examine a thing (entering into it) and the extrinsic one, (dealing with it from one perspective or another by turning around it) . It is worthy of note that the intrinsic way corresponds to the formal procedure associated with “trans-subjectivity” where the different perspectives are eliminated by compensation, while the extrinsic way has to be linked to subjectivity and /or “inter-subjectivity”, where the subjective measurements are not eliminated but on the contrary underlined and specified. These considerations are absent from all other formulations of motion, simply because the underlying logic is the binary one, opposing “truthfulness” to “falsity” in an absolute manner, and not relatively to one point of view or another.

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